Learning Tests - Basic terms

Nguyen Thi Tu Van[1] Nguyen Quang Vinh[2]

[1]University of Medicine and Pharmacy of Ho Chi Minh City

[2]Nguyen Tri Phuong Hospital

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Outline

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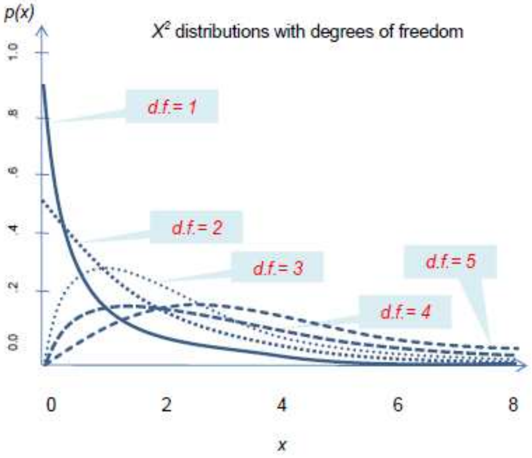
Chi-square (χ^2)

- One of the most widely used directly or indirectly.
- Testing hypothesis where data in form of category.
- To test differences between proportions.

Chi-square distributions
with different degrees of freedom



Outline

Applications of the χ^2 Statistic

The principle of this calculation is to compare observed frequencies (OBSERVATION) to expected frequencies (HYPOTHESIS)

- (1) Test of goodness-of-fit
- (2) Test of independence
- (3) Test of homogeneity *(test of independence with fixed marginal totals)*

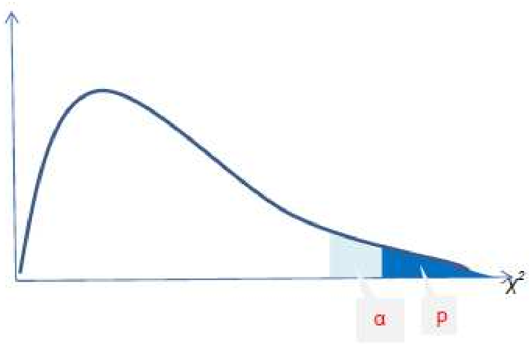
χ^2 test of goodness-of-fit

- A two-tailed test on p
- For Binomial situation:
  - $H_O : p = p_0$
  - $H_A : p \neq p_0$
- For Multinomial situation:
  - $H_O : p_1 = p_{1_0},\ p_2 = p_{2_0},\ \ldots,\ p_k = p_{k_0}$
  - H_A : at least one of the $p_i' s$ is incorrect

χ^2 test of goodness-of-fit

Test statistic: $\chi_c^2 = \sum \frac{(O-E)^2}{E}$
df= number of categories - 1
reject H_O if $\chi_c^2 > \chi_{\alpha,df}^2$

Rejection area



χ^2 test of goodness-of-fit (opt.)

Testing H_O: $p = p_0$ vs. H_A: $p \neq p_0$

- Using z test:
  $z_c = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$
  reject H_O if $|z_c| > z_{1-\frac{\alpha}{2}}$
- Using χ^2 test:
  $\chi^2 = \sum \frac{(O-E)^2}{E} = \frac{(O_1 - E_1)^2}{E_1} + \frac{(O_2 - E_2)^2}{E_2}$
  reject H_O if $\chi^2 > \chi^2_{\alpha, df=1}$

χ^2 test of goodness-of-fit

- How well the distribution of sample data conforms to some theorical distribution*
- d.f. = k − r
- Small expected frequencies: there is disagreement among writers: 10, 5, 1 (*Cochran*).
  - Combining adjacent categories to achieve the suggested minimum.
  - When combining →↓number of categories→↓d.f.

χ^2 test of independence

- Most frequent use of χ^2 distribution
- A ***single*** population, where each member was classified according to 2 criteria:
  1^{st} *criteria* : row
  2^{nd} *criteria* : column
- Contingency table: r rows, c columns
- H_O : 2 criteria of classification are independent
  H_A : 2 criteria of classification are not independent
- df = (r − 1)(c − 1)

χ^2 test of independence

Small expected frequencies

- Small expected frequencies:
  df > 2 & no more than 20% of expected frequencies < 5 →1
  df < 30 → 2
  $n \geq 40$ → 1
- χ^2 test should not be used if:
  $n < 20$, or
  $20 \leq n < 40$ & any $E_i < 5$

χ^2 test of independence

$$\chi^2 = \sum \frac{(O-E)^2}{E} = \frac{(O_1-E_1)^2}{E_1} + \frac{(O_2-E_2)^2}{E_2} + \ldots$$

reject H_O if $\chi^2_c > \chi^2_{\alpha, df=(r-1)(c-1)}$

χ^2 test of independence

For the 2 x 2 contingency table:

$$\chi^2 = \sum \frac{(O-E)^2}{E} = \frac{(O_1-E_1)^2}{E_1} + \frac{(O_2-E_2)^2}{E_2}$$

$$\chi^2 = \frac{n(ad-bc)^2}{(a+b)(a+c)(b+d)(c+d)}$$

reject H_O if $\chi^2 > \chi^2_{\alpha,df=(r-1)(c-1)=1}$

χ^2 test of independence (opt.)

Yates' correction (also called continuity correction) - Pro and Cons

$$\chi^2_{corrected} = \sum \frac{(|O-E|-.5)^2}{E} = \frac{(|O_1-E_1|-.5)^2}{E_1} + \frac{(|O_2-E_2|-.5)^2}{E_2}$$

$$\chi^2_{corrected} = \frac{n(|ad-bc|-.5n)^2}{(a+b)(a+c)(b+d)(c+d)}$$

reject H_O if $\chi^2_{corrected} > \chi^2_{\alpha,df=(r-1)(c-1)=1}$

χ^2 test of homogeneity

χ^2 test of independence with Fixed Marginal Totals

- To determine whether the distinct populations can be viewed as belonging to the same population (in terms of the criteria).

χ^2 test of homogeneity vs. χ^2 test of independence

- χ^2 test of independence: row and column totals are not under the control of the investigator
  χ^2 test of homogeneity: either row or column totals may be under the control of the investigator
- χ^2 test of independence: ? independent (the 2 criteria)
  χ^2 test of homogeneity: ? homogeneous (the samples drawn from the same population
- χ^2 test of homogeneity & χ^2 test of independence are mathematically equivalent but conceptually different.

χ^2 test of homogeneity (opt.)

- χ^2test of Homogeneity for the 2-sample case provides an alternative method for testing the H_O that: 2 population proportions are equal.
- A method for comparing of 2 population proportions using z statistic with pool proportion($\bar{p}$):
  Let $\hat{p}_1 = \frac{x_1}{n_1}$; $\hat{p}_2 = \frac{x_2}{n_2}$; $\bar{p} = \frac{x_1+x_2}{n_1+n_2}$
  Test statistic: $z_c = \frac{(\hat{p}_1-\hat{p}_2)-(p_1-p_2)_0}{\sqrt{\frac{\bar{p}(1-\bar{p})}{n_1}+\frac{\bar{p}(1-\bar{p})}{n_2}}}$
- Note: $z_c = \frac{(\hat{p}_1-\hat{p}_2)-(p_1-p_2)_0}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1}+\frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}}$ is just for discussion purposes only. This equation should never be used as the test statistic for the difference between 2 proportions.

Outline

Fisher's exact test

- When expected value in χ^2test statistic is small.

	Treatment	Control	**Total**
O+	x	$K - x$	**K**
O-	$n - x$	$(N\text{-}K)\text{-}(n\text{-}x)$	**N - K**
Total	**n**	**N-n**	**N**

$$N \rightarrow \left\{ \begin{matrix} K & & x \\ & & \\ N-K & & n\text{-}x \end{matrix} \right\} \leftarrow n$$

$$P(x) = \frac{{}_K C_x \cdot {}_{N-K} C_{n-x}}{{}_N C_n}$$

Example

We have a result from a trial as follow:

	Treatment	Control	Total
O+	6	1	**7**
O-	2	4	**6**
Total	**8**	**5**	**13**

Listing all possible tables in the sample of size 13, which have:
7 positive outcomes & 8 subjects in treatment group
$\rightarrow$ We have 6 tables as follow:

	Treatment	Control	Total
O+	*7*	0	**7**
O-	1	5	**6**
Total	**8**	**5**	**13**

$$P(x=7) = \frac{{}_7C_7 \cdot {}_6C_1}{{}_{13}C_8} = \frac{6}{1287} = .0047$$

	Treatment	Control	Total
O+	*6*	1	**7**
O-	2	4	**6**
Total	**8**	**5**	**13**

$$P(x=6) = \frac{{}_7C_6 \cdot {}_6C_2}{{}_{13}C_8} = .0816$$

	Treatment	Control	Total
O+	*5*	2	**7**
O-	3	3	**6**
Total	**8**	**5**	**13**

$$P(x=5) = \frac{{}_7C_5 \cdot {}_6C_3}{{}_{13}C_8} = .3262$$

	Treatment	Control	Total
O+	*4*	3	**7**
O-	4	2	**6**
Total	**8**	**5**	**13**

$$P(x=4) = \frac{{}_7C_4 \cdot {}_6C_4}{{}_{13}C_8} = .4070$$

	Treatment	Control	Total
O+	*3*	4	**7**
O-	5	1	**6**
Total	**8**	**5**	**13**

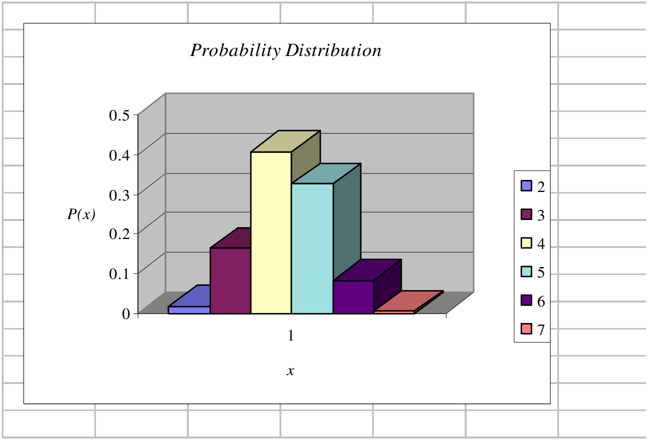
$$P(x=3)=\frac{{}_7C_3 \cdot {}_6C_5}{{}_{13}C_8} = .1632$$

	Treatment	Control	Total
O+	*2*	5	**7**
O-	6	0	**6**
Total	**8**	**5**	**13**

$$P(x=2)=\frac{{}_7C_2 \cdot {}_6C_6}{{}_{13}C_8} = .0163$$

A useful check is that all the probabilities should sum to one (within the limits of rounding)

Probability distribution



Hypothesis

- $H_O : \pi_T = \pi_C$
  (no difference between treatment & control group)
- $H_A : \pi_T > \pi_C$ (1-tailed), or
- $H_A : \pi_T \neq \pi_C$ (2-tailed)

Calculate p value

- The observed set has a probability of 0.0816
- The p value is the probability of getting the observed set, or one more extreme.
- One tailed p value:
  - (1) p(x≥6)= p(x=6) + p(x=7)= 0.0816 + 0.0047= 0.0863 (this is the conventional approach).
  - (2) Armitage & Berry (1994) favor the mid p value: 0.5 x 0.0816 + 0.0047 = 0.0455
- Two tailed p value:
  - (1) p(x≥6 or x≤2)= p(x=2)+ p(x=6)+ p(x=7) = 0.0816 + 0.0047 + 0.0163 = 0.1026
  - (2) Double the one tailed result *(approximation)*, thus: p= 2 x 0.0863 = 0.1726 (for the conventional) or p= 2 x 0.0455 = 0.091 (for the mid P approach)

Outline

The conventional vs. the mid p (opt.)

- The conventional approach to calculating the p value for Fisher's exact test has been shown to be conservative (that is, it requires more evidence than is necessary to reject a false H_O)
- The mid P is less conservative (that is more powerful) & also has some theoretical advantages

Why is Fisher's test called an exact test? (opt.)

- Because of the discrete nature of the data, and the limited amount of it, combinations of results which give the same marginal totals can be listed, and probabilities attached to them.
  $\rightarrow$ thus, given these marginal totals we can work out exactly what is the probability of getting an observed result.

The t distribution

◇ PROBLEM:

- σ is known & not known μ (!)
- Indeed, it is the usual case, σ & μ is unknown

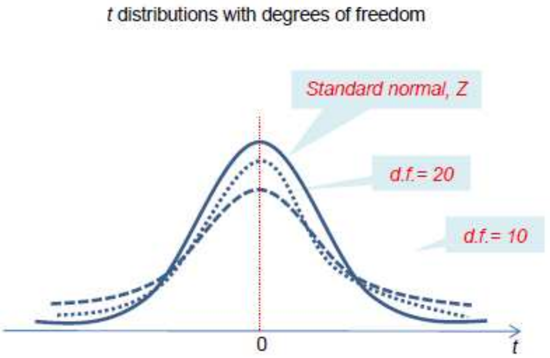
◇ We cannot make use the statistic: $z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$ because σ is unknown, even when n is large,

$\rightarrow$ use $s = \sqrt{\frac{\sum(x_i - \overline{x})^2}{n-1}}$ to replace σ

The t distribution

- William Sealy Gosset "Student" (1908) → Student's t distribution = t distribution.
- The quantity: $t = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}$ follows this distribution.

t distributions with degrees of freedom



Notice

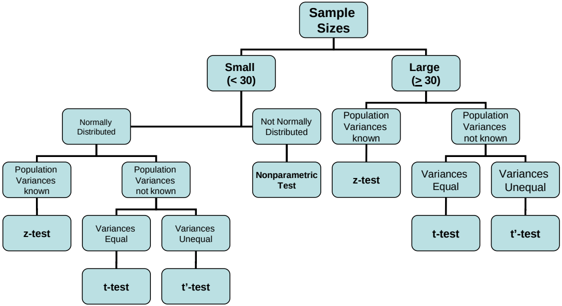
A requirement for valid use of the t distribution:
☼ sample must be drawn from a normal distribution.
☼ an assumption of, at least, a mound-shaped population distribution be tenable.

An interval estimate

- In general, an interval estimate is obtained by the formula: *estimator ± (reliability coefficient) x (standard error)*
- What is different is the source of the reliability coefficient:
  - In particular, when sampling is from a normal distribution with known variance, an interval estimate for μ may be expressed as: $\overline{x} \pm z_{\frac{\alpha}{2}} \sigma_{\bar{x}}$
  - when sampling is from a normal distribution with unknown variance, the $100(1-\alpha)\%$ confidence interval estimate for the population mean, μ, is given by: $\overline{x} \pm t_{\frac{\alpha}{2},n-1} \frac{s}{\sqrt{n}}$

z, t or t'

Deciding between z, t, or t'



Flowchart for use in deciding whether the reliability factor should be use z, t, or t' when making inferences about the difference between two population means (* use a nonparametric procedure)

The Mann-Whitney Test

- When small samples from suspected nonnormal population - substitution of 2-sample t test.
- Assumptions for M-W test:
  1. Independent, Random samples
  2. Data at least ordinal
  3. If the 2 populations differ, they differ only in location (e.g., the 2 populations have the same variance and shape).
- Hypothesis: H_O : 2 populations have identical of the probability distribution *vs.*
  H_A : 2 populations differ in location (2-tailed), or
  H_A : population 1 is shifted to the right of population 2 (1-tailed), or
  H_A : population 2 is shifted to the right of population 1 (1-tailed)

Correlation & Regression

- Nature & strength of the relationship between 2 variables: BP & age, cholesterol & age, size & weight of fetus...
  $\rightarrow$ Correlation & Regression analysis

Correlation & Regression

- Correlation (1888): the strength of the associative relationship between 2 variables.
  Correlation refers to the interdependence or co-relationship of variables.
- Regression (1899): the causal relationship between variables: predict, or estimate.
  Regression is a way of describing how one variable, the outcome, is numerically related to predictor variable(s).

Data types for correlation/regression analysis

- Need our data to be quantitative / numerical / continuous.
- If data can meaningfully be portrayed on a scatter plot.

Outline

1. Chi-square
   - Chi-square distribution & The analysis of frequencies
   - Applications of the χ^2 Statistic
2. Fisher
   - Calculating p value in Fisher's exact test
   - The conventional vs. the mid p (opt.)
3. Student's t
4. Mann-Whitney U
5. Pearson and Spearman's correlation
   - Pearson and Spearman correlation
   - Cause and Effect
   - Correlation vs. Linear Regression

Pearson correlation coefficient

- Pearson's correlation coefficient is a measure of the closeness linear association between X and Y.
- Denoted by r (sample statistic), and ρ (population parameter).

Interpretation of r

r is a much abused statistic

- r takes values between –1 and +1 inclusive: -1 < r < 1. Sign of + or –.
- Value r doesn't mean the steepness of the slope.
- The large |r| is, the stronger is the linear relationship.
  + Values of r close to –1 or +1 indicate a strong (negative or positive) linear relationship.
  r is close to ± 1 then this does NOT mean that there is a good causal relationship between X and Y. It shows only that the sample data is close to a straight line.
  + Values of r close to zero indicate little linear relationship between 2 variables.
  Even if r close to zero, there still may be a strong relationship in the form of a curve.

Interpretation of r *(opt.)*

r is a much abused statistic

- Assumption of Pearson's correlation: *at least* one variable must follow a normal distribution.
- Confidence limits are constructed for r using Fisher's z-transformation.
- r^2 is closest to 1 when n = k + 1.
    - But n should be $\geq 3(k+1)$ for a more reliable regression model.

Significance Test for Pearson's Correlation

$H_O : \rho = 0$ (There is no linear relationship)
$H_A : \rho \neq 0$ (There is a linear relationship)

- The $H_O : \rho = 0$ is evaluated using modified t-test.
- Conclusion – significant linear correlation (i.e. $\rho \neq 0$) if p-value < 0.05

What if our data are only non-linearly related??

- Techniques are available for some non-linear relationships, e.g. Spearman's correlation coefficient can detect relationships which are (at least) monotonic.

Outline

Cause and Effect

- Evidence of correlation does not (necessarily) mean that a cause and effect relationship exists.

Outline

Correlation vs. Linear Regression

- *Only if* substantive theory *(i.e. the science)* suggests a causal relationship between 2 variables do we have grounds to use linear regression analysis.
- Otherwise, correlation analysis is all we can use.
  - i.e. We are restricted to talking about associative relationships.
- Cross-sectional studies???
- Although mathematically similar, Regression analysis represents a much more rigorous formulation of the relationships between variables.
  It also introduces: a statistical model
  - Models are the mathematical representation (and simplification) of the system under study.

The Use of Linear Regression

(If we believe that 2 variables do exhibit an underlying linear pattern)

- Data description, assessment:
  - Quantify the relationship between variables
- Parameter estimation
- Prediction and estimation:
  - Make prediction and validate the test
- Control: *(in the case of multiple variables)*
  - Adjusting for confounding effect

What is a good value of R^2

It depends on the area of application:

- In the biological and social sciences, variables tend to be more weakly correlated and there is a lot of noise. $R^2$is expected low in these areas - a value of 0.6 might be considered good.
- In physics and engineering, most data comes from closely controlled experiments, it is expected to get much higher R^2 and a value of 0.6 would be considered low.
  $\Rightarrow$*Experience with the particular area is necessary to judge R^2*.

Quiz

- Can we arrange the equation: $\hat{Y} = b_0 + b_1 X$
  say, $\hat{X} = -\frac{b_0}{b_1} + \frac{\hat{Y}}{b_1}$
- The answer is *no*,
  although the coefficient of correlation, r, is the same in either case.

Summary

- Chi-square: the analysis of frequencies: goodness-of-fit, independence, homogeneity.
- Fisher's exact test: using when expected value in χ^2test statistic is small.
- t test: a family of distributions, approaches the normal distribution as $n-1$ approaches infinity.
- Mann-Whitney test: when the assumptions for using the independent t test are violated.
- Correlation: associative relationship
  - Pearson correlation for linear relationships.
  - Spearman correlation for non-linear - (at least) monotonic relationships.
- Regression: causal relationship.

Exercises

1. Why should not we use Chi-square distribution when expected value in χ^2test statistic is small?
2. What happen if we violate assumption(s) for using a test?