Continuous Limit of Multiple Gravitational Lens Effect and the Optical Scalar Equations*

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Abstract

We study the continuous limit of the multiple gravitational lensing theory based on the thin lens approximation. Under this assumption, we define an angular diameter distance which depends on the light-path as $\tilde{d} = \mu^{-1/2} d_{\rm DR}$ (μ and $d_{\rm DR}$ denote the gravitational magnification factor and the Dyer-Roeder distance). We also show that the distance satisfies the optical scalar equation in an inhomogeneous universe. Our formalism yields the relation between quantities (convergence, shear, and twist) in the gravitational lensing theory and those (rates of expansion, shear and rotation) in the scalar optics theory.

Keywords: cosmology: theory —— distance measure —— optical scalar equations —— gravitational lensing

1. Introduction

In a general space, the ray-bundle from a source obeys the optical scalar equations (Sachs 1961):

$$\frac{d}{d\tau}\Theta + \Theta^2 + |\Sigma|^2 = \mathcal{R}, \quad \frac{d}{d\tau}\Sigma + 2\Theta\Sigma = \mathcal{F}, \quad \text{and} \quad \frac{d}{d\tau}\omega + 2\Theta\omega = 0, \tag{1}$$

where Θ, Σ, and ω denote the expansion-, shear-, rotation-rate, and τ is the affine parameter of the null geodesic. In eq. (1) $\mathcal{R}$ and $\mathcal{F}$ are the Ricci term and the Weyl term, respectively. The expansion rate is expressed in terms of the cross-sectional area $\mathcal{A}$ or the angular diameter distance D:

$$\Theta = \frac{1}{2}\frac{d}{d\tau}\ln\mathcal{A} = \frac{d}{d\tau}\ln D. \tag{2}$$

Then the first equation of (1) is written as:

$$\frac{d^2}{d\tau^2}D + \left(|\Sigma|^2 - \mathcal{R}\right)D = 0. \tag{3}$$

Homogeneous universe

In a Fiedmann-Lemaître universe model (FL model), the angular diameter distance from an observer to a source at a redshift z is given by the Mattig formula:

$$D_{\rm FL}(0;z) = \frac{c}{H_0}d_{\rm FL}(0;z) = \frac{(c/H_0)}{\sqrt{K}(1+z)}\sin\left[\sqrt{K}\int_0^z\frac{dz'}{Y(z')}\right], \tag{4}$$

where $Y(z) = \sqrt{\Omega_0(1+z)^3 + \lambda_0 - K(1+z)^2}$, $K = \Omega_0 + \lambda_0 - 1$, and Ω_0 & λ_0 are the present values of the density parameter and of the cosmological constant. $d_{\rm FL}$ satisfies the first equation of (1) as follows:

$$\frac{d^2}{d\tau^2}d_{\rm FL}(0;z) + \frac{3}{2}\Omega_0(1+z)^5 d_{\rm FL}(0;z) = 0. \tag{5}$$

In this model, the shear-rate automatically vanishes.

Inhomogeneous universe

The Universe is, on average, described by an FL model, but locally inhomogeneous. **The clumpy model** may be able to be adopted, where it is assumed that the matter density with fraction $\bar{\alpha}$ of the mean density of the universe is smoothly distributed and the rest matters are concentrated in clumps. Under this assumption the angular diameter distance $D_{\rm DR}(0;z)$ is given by the Dyer-Roeder distance (Dyer & Roeder 1972,1973) if the ray-bundle propagates in an evacuate tube away from all the clumps, i.e. "the shear free assumption $\Sigma \cong 0$". Then $D_{\rm DR}=(c/H_0)d_{\rm DR}$ satisfies the following equation:

$$\frac{d^2}{d\tau^2}d_{\rm DR}(0;z) + \frac{3}{2}\bar{\alpha}\Omega_0(1+z)^5 d_{\rm DR}(0;z) = 0. \tag{6}$$

The effect of clumps?

The light-ray passing near/through a clump is gravitationally lensed. In this case, the observed flux $f_{\rm obs.}$ is magnified by factor μ (the gravitational magnification factor).

$$f_0 = \frac{L}{4\pi(1+z)^4(c/H_0)^2 d_{\rm DR}^2(0;z)} \xrightarrow[\text{lensed}]{\text{gravitationally}} f_{\rm obs.} = \frac{\mu L}{4\pi(1+z)^4(c/H_0)^2 d_{\rm DR}^2(0;z)} \tag{7}$$

In a case of $\mu>1$, the source of the light-ray is observed as a nearer object. Then the apparent angular diameter distance $\tilde{d}$ is written as $\tilde{d} = \mu^{-1/2}d_{\rm DR}$.

Now we have a problem whether the distance really satisfies the optical scalar equations for the light-ray passing near/through clumps or not.

2. Multiple Gravitational Lens Effect

The light-ray propagates in an inhomogeneous universe. Each clump intervening between the source and us may more or less affect on the light-ray. Hence we must consider the multiple gravitational scattering of the light-ray. Here we investigate the multiple gravitational scattering of the light-ray by taking into account the multiple gravitational lensing.

We assume that there are N lenses which are randomly distributed at redshift z_i ($0 \le z_1 \le z_2 < \ldots \le z_N$) between the source at redshift $z_{\rm S}(=z_{N+1})$ and an observer. The origin of each lens plane is located on the point intersected with the line of sight (see Figure 1). The multiple lens equations is given by

$$\boldsymbol{\theta}_{\rm S} = \boldsymbol{\theta}_1 - \sum_{k=1}^{N}\frac{d_{\rm DR}(z_k;z_{\rm S})}{d_{\rm DR}(0;z_{\rm S})}\boldsymbol{\alpha}_k(\boldsymbol{\theta}_k), \quad \boldsymbol{\theta}_k = \boldsymbol{\theta}_1 - \sum_{i=1}^{k-1}\frac{d_{\rm DR}(z_i;z_k)}{d_{\rm DR}(0;z_k)}\boldsymbol{\alpha}_i(\boldsymbol{\theta}_i), \tag{8}$$

where $\boldsymbol{\theta}_i$ and $\boldsymbol{\theta}_{\rm S}$ are the angular positions of the light ray on the i-th lens plane and of the source plane, $\boldsymbol{\alpha}_i$ and $d_{\rm DR}(z_i;z_k)$ denote the deflection angle on the i-th lens plane and the Dyer-Roeder distance from the i-th lens plane to the k-th lens plane.

According to Schneider, Ehlers & Falco (1992), we can define a useful function, i.e. $\chi(z)$ function as follows:

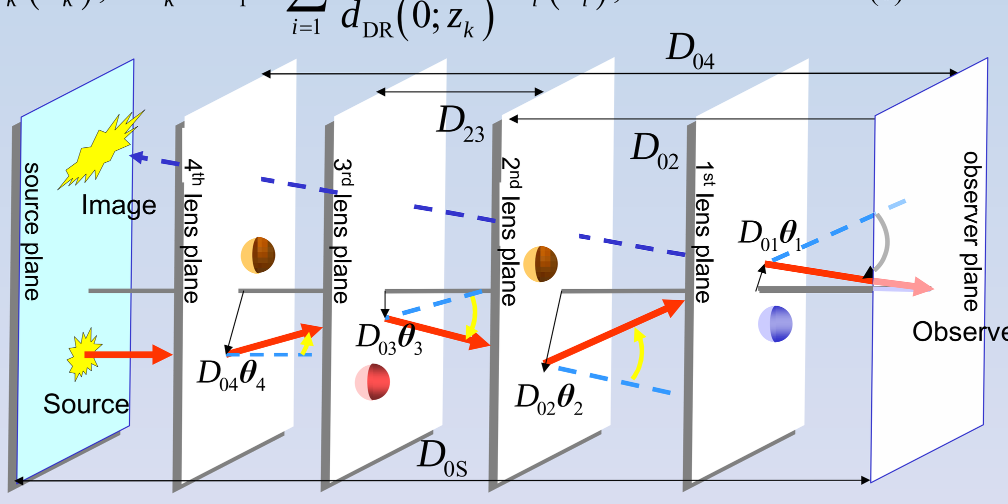

Figure 1. Geometry of multiple gravitational lensing system

$$\chi(z_i) = \chi_i = \int_{z_i}^{\infty}\frac{dz}{(1+z)^2 d_{\rm DR}^2(0;z)Y(z)} = \int_{\tau(z_i)}^{\infty}\frac{d\tau}{d_{\rm DR}^2(0;z)}. \tag{9}$$

This function relates to the Dyer-Reoder distance as

$$d_{\rm DR}(z_i;z_j) = (1+z_i)d_{\rm DR}(0;z_i)d_{\rm DR}(0;z_j)(\chi_i - \chi_j). \tag{10}$$

By substituting this relation (10) to eq. (8) we can obtain the following difference equations

$$\boxed{\frac{\boldsymbol{\theta}_{i+1}-\boldsymbol{\theta}_i}{\chi_i - \chi_{i+1}} - \frac{\boldsymbol{\theta}_i - \boldsymbol{\theta}_{i-1}}{\chi_{i-1}-\chi_i} = -(1+z_i)d_{\rm DR}(0;z_i)\boldsymbol{\alpha}_i(\boldsymbol{\theta}_i),} \tag{11}$$

where

$$\boldsymbol{\alpha}_i(\boldsymbol{\theta}_i) = \frac{4G}{H_0^2}\frac{d_{\rm DR}(0;z_i)\Delta z_i}{(1+z_i)Y(z_i)}\iint_D d^2\boldsymbol{\varphi}\,\delta_{\bar{\alpha}}\rho\left(D_{\rm DR}(0;z)\boldsymbol{\varphi}, Z(z)\right)\frac{\boldsymbol{\theta}_i - \boldsymbol{\varphi}}{|\boldsymbol{\theta}_i - \boldsymbol{\varphi}|^2} \tag{12}$$

is a deflection angle on the i-th lens plane due to a clump with mass density

$$\delta_{\bar{\alpha}}\rho\left(D_{\rm DR}(0;z)\boldsymbol{\varphi}, Z(z)\right) \equiv \rho\left(D_{\rm DR}(0;z)\boldsymbol{\varphi}, Z(z)\right) - \bar{\alpha}\bar{\rho}(z)\,(\ge 0). \tag{13}$$

Now we can take a limit of $\Delta z_i\left(= z_{i+1} - z_i\right) \to 0$ with $\bar{\rho}_0$ fixed!
(Go "§3. Continuous limit"!)

∗ This presentation is based on Yoshida, Nakamura & Omote (2004).

3. Continuous Limit and Optical Scalar Equations

In eq. (11) we take the limit of $\Delta z_i = z_{i+1} - z_i \to 0$ (i.e., $\Delta\chi_i = \chi_{i+1} - \chi_i \to 0$) to obtain a second differential equation of $\boldsymbol{\theta}$ with respect to χ as follows:

$$\frac{d^2\boldsymbol{\theta}}{d\chi^2} \equiv \boldsymbol{\theta}'' = -\frac{4G}{H_0^2}(1+z)^2 d_{\rm DR}^4(0;z)\iint_D d^2\boldsymbol{\varphi}\,\delta_{\bar{\alpha}}\rho\left(D_{\rm DR}(0;z)\boldsymbol{\varphi}, Z\right)\frac{\boldsymbol{\theta}-\boldsymbol{\varphi}}{|\boldsymbol{\theta}-\boldsymbol{\varphi}|^2}. \tag{14}$$

The prime $'$ denotes the derivative with respect to χ. The solution of this equation with initial conditions $\boldsymbol{\theta}(0) = \boldsymbol{\theta}_0$ and $\boldsymbol{\theta}'(0) = \mathbf{0}$ yields a map from $\boldsymbol{\theta}_0$ on the observer plane to $\boldsymbol{\theta}(z)$ on the lens plane at redshift z. Then we can define the Jacobian matrix $\mathbf{A}$ ($=\partial\boldsymbol{\theta}/\partial\boldsymbol{\theta}_0$) for this map which satisfies an equation similar to eq. (14):

$$\frac{d^2}{d\chi^2}\mathbf{A} \equiv \mathbf{A}'' = -(1+z)^2 d_{\rm DR}^4(0;z)\begin{pmatrix}\frac{4\pi G}{H_0^2}\delta_{\bar{\alpha}}\rho - \gamma_x(\boldsymbol{\theta}) & -\gamma_y(\boldsymbol{\theta}) \\ -\gamma_y(\boldsymbol{\theta}) & \frac{4\pi G}{H_0^2}\delta_{\bar{\alpha}}\rho + \gamma_x(\boldsymbol{\theta})\end{pmatrix}\mathbf{A} \equiv -\mathbf{W}''\mathbf{A}, \tag{15}$$

where $\boldsymbol{\gamma} = (\gamma_x, \gamma_y)$ is the shear due to a clump at redshift z given by

$$\gamma_x(\boldsymbol{\theta}, z) = \frac{4G}{H_0^2}\iint_D d^2\boldsymbol{\varphi}\,\delta_{\bar{\alpha}}\rho\left(D_{\rm DR}(0;z)\boldsymbol{\varphi}, Z(z)\right)\frac{(\theta_x - \varphi_x)^2 - (\theta_y - \varphi_y)^2}{|\boldsymbol{\theta}-\boldsymbol{\varphi}|^4},$$
$$\gamma_y(\boldsymbol{\theta}, z) = \frac{8G}{H_0^2}\iint_D d^2\boldsymbol{\varphi}\,\delta_{\bar{\alpha}}\rho\left(D_{\rm DR}(0;z)\boldsymbol{\varphi}, Z(z)\right)\frac{(\theta_x - \varphi_x)(\theta_y - \varphi_y)}{|\boldsymbol{\theta}-\boldsymbol{\varphi}|^4}. \tag{16}$$

The Jacobian matrix A:

In general, the Jacobian matrix $\mathbf{A}$ is given as

$$\mathbf{A} \equiv \begin{pmatrix} K_x + G_x & K_y + G_y \\ -K_y + G_y & K_x - G_x\end{pmatrix}, \tag{17}$$

where $\boldsymbol{K}(\chi(z)) = (K_x(\chi(z)), K_y(\chi(z)))$ and $\boldsymbol{G}(\chi(z)) = (G_x(\chi(z)), G_y(\chi(z)))$ denote the cumulative convergence-, twist-, and shear-terms, respectively, whose initial conditions are $\boldsymbol{K}(\chi(0)) = (1, 0)$, $\boldsymbol{G}(\chi(0)) = \mathbf{0}$ and $\boldsymbol{K}'(\chi(0)) = \boldsymbol{G}'(\chi(0)) = \mathbf{0}$. In terms of the elements of $\mathbf{A}$, therefore we can express the gravitational magnification factor as $\mu(\boldsymbol{\theta}(z)) = \det\mathbf{A}^{-1} = 1/(\boldsymbol{K}^2 - \boldsymbol{G}^2)$, which depends on the light-path $\boldsymbol{\theta}(z)$ (solution of eq.[14]). Hereafter, we restrict ourselves to the case of $\det\mathbf{A} \neq 0$.

As mentioned §1, we define an apparent angular diameter distance as follows:

$$\boxed{\tilde{d}\left(\boldsymbol{\theta}(z)\,|\,0;z\right) = \mu^{-1/2}\left(\boldsymbol{\theta}(z)\right)d_{\rm DR}(0;z).} \tag{18}$$

Moreover we define some quantities $\tilde{\Theta}$, $\tilde{\Sigma}$ and $\tilde{\omega}$ in terms of $\tilde{d}$ and elements of $\mathbf{A}$ as follows:

$$\tilde{\Theta} \equiv \frac{d}{d\tau}\ln\tilde{d}\left(\boldsymbol{\theta}(z)\,|\,0;z\right), \tilde{\Sigma} \equiv -(\delta_{ab} + i\varepsilon_{ab})\frac{\mu(K_a G_b' - K_a' G_b)}{d_{\rm DR}^2(0;z)}, \quad \tilde{\omega} \equiv -\frac{\varepsilon_{ab}\mu(K_a K_b' + G_a G_b')}{d_{\rm DR}^2(0;z)}. \tag{19}$$

By using these quantities and a matrix $\mathbf{A}'\mathbf{A}^{-1}$:

$$\mathbf{A}'\mathbf{A}^{-1} = -d_{\rm DR}^2(0;z)\begin{pmatrix}\tilde{\Theta} - \Theta_{\rm DR} + {\rm Re}[\tilde{\Sigma}] & {\rm Im}[\tilde{\Sigma}] + \tilde{\omega} \\ {\rm Im}[\tilde{\Sigma}] - \tilde{\omega} & \tilde{\Theta} - \Theta_{\rm DR} - {\rm Re}[\tilde{\Sigma}]\end{pmatrix}, \quad \Theta_{\rm DR} = \frac{d}{d\tau}\ln d_{\rm DR}(0;z), \tag{20}$$

we can rewrite eq.(15) as follows

$$\begin{aligned}\mathbf{A}''\mathbf{A}^{-1} &= \left(\mathbf{A}'\mathbf{A}^{-1}\right)' + \left(\mathbf{A}'\mathbf{A}^{-1}\right)^2 \\ &= d_{\rm DR}^4(0;z)\left[\frac{d}{d\tau}\begin{pmatrix}\tilde{\Theta} - \Theta_{\rm DR} + {\rm Re}[\tilde{\Sigma}] & {\rm Im}[\tilde{\Sigma}] + \tilde{\omega} \\ {\rm Im}[\tilde{\Sigma}] - \tilde{\omega} & \tilde{\Theta} - \Theta_{\rm DR} - {\rm Re}[\tilde{\Sigma}]\end{pmatrix}\right. \\ &\quad \left. + \begin{pmatrix}\tilde{\Theta}^2 + |\tilde{\Sigma}|^2 - \tilde{\omega}^2 - \Theta_{\rm DR}^2 + 2\tilde{\Theta}{\rm Re}[\tilde{\Sigma}] & 2\tilde{\Theta}\left({\rm Im}[\tilde{\Sigma}] + \tilde{\omega}\right) \\ 2\tilde{\Theta}\left({\rm Im}[\tilde{\Sigma}] - \tilde{\omega}\right) & \tilde{\Theta}^2 + |\tilde{\Sigma}|^2 - \tilde{\omega}^2 - \Theta_{\rm DR}^2 - 2\tilde{\Theta}{\rm Re}[\tilde{\Sigma}]\end{pmatrix}\right] = -\mathbf{W}''.\end{aligned} \tag{21}$$

Eqs. (6), (13) and (21) yield the following equations of $\tilde{\Theta}$, $\tilde{\Sigma}$ and $\tilde{\omega}$:

$$\frac{d}{d\tau}\tilde{\Theta} + \tilde{\Theta}^2 + |\tilde{\Sigma}|^2 - \tilde{\omega}^2 = -\frac{4\pi G}{H_0^2}(1+z)^2\rho\left(D_{\rm DR}(0;z)\boldsymbol{\theta}, Z(z)\right), \tag{22}$$

$$\frac{d}{d\tau}\tilde{\Sigma} + 2\tilde{\Theta}\tilde{\Sigma} = (1+z)^2(\gamma_x + i\gamma_y), \tag{23}$$

$$\frac{d}{d\tau}\tilde{\omega} + 2\tilde{\Theta}\tilde{\omega} = 0. \tag{24}$$

The last equation (24) shows that $\tilde{\omega}$ vanishes. Moreover we found that the right hand side of eq. (22) can be regarded as the Ricci term, $\mathcal{R}$. The right hand side of eq. (23) is also rewritten as

$$(1+z)^2(\gamma_x + i\gamma_y) = \frac{4G}{H_0^2}(1+z)^2\iint_D d^2\boldsymbol{\varphi}\frac{\delta_{\bar{\alpha}}\rho\left(D_{\rm DR}(0;z)\boldsymbol{\varphi}, Z(z)\right)}{(\theta^* - \varphi^*)^2}, \tag{25}$$

where $\theta^* = \theta_x - i\theta_y$, $\varphi^* = \varphi_x - i\varphi_x$. Eq. (25) slightly differs from the Weyl term obtained in the perturbation theory of the general relativity. The difference between them comes from the thin lens approximation adopted in this presentation. Then we can regard eq. (25) as the Weyl term $\mathcal{F}$ as long as the approximation is valid.

Finally we have obtained the equations of $\tilde{\Theta}$, $\tilde{\Sigma}$ and $\tilde{\omega}$:

$$\boxed{\frac{d}{d\tau}\tilde{\Theta} + \tilde{\Theta}^2 + |\tilde{\Sigma}|^2 = \mathcal{R}, \quad \frac{d}{d\tau}\tilde{\Sigma} + 2\tilde{\Theta}\tilde{\Sigma} = \mathcal{F} \quad \text{and} \quad \frac{d}{d\tau}\tilde{\omega} + 2\tilde{\Theta}\tilde{\omega} = 0,} \tag{26}$$

which we can regard as the optical scalar equations (1) under the thin lens approximation.

4. Conclusions

We studied the continuous limit of the multiple gravitational lensing theory under the thin lens approximation. In this limit, we found that $\tilde{\Theta}$, $\tilde{\Sigma}$ and $\tilde{\omega}$ constructed from elements of the Jacobian matrix can be regarded as the expansion-, shear- and rotation rates in the scalar optics theory. Since $\tilde{d}$ satisfies the optical scalar equation in a general space, then, we can adopt it as the angular diameter distance to a specific object.

References

Dyer, C.C. & Roeder, R.C. 1972, ApJ,174,L115
Dyer, C.C. & Roeder, R.C. 1973, ApJ,180,L21
Sachs, R.K. 1961, Proc.Roy.Soc.London A, 264,727
Schneider, P., Elhers, J. & Falco, E.E. 1992, Gravitational Lenses, (Springer-Verlag)
Yoshida, H., Nakamura, K. & Omote, M. 2004 submitted (astro-ph/0407040)