Continuous Limit of Multiple Gravitational Lens Effect and the Optical Scalar Equations*

Hiroshi Yoshida
Department of Physics, Fukushima Medical University, 1 Hikari-ga-oka, Fukushima-City 960-1295, Japan
yoshidah@fmu.ac.jp

Abstract
We study the continuous limit of the multiple gravitational lensing theory based on the thin lens approximation. Under this assumption, we define an angular diameter distance which depends on the light-path as \( d = \mu'' \delta d_{\text{obs}}(\mu, \Delta u) \) and denote the gravitational magnification factor and the Dyer-Roeder distance. We also show that the distance satisfies the optical scalar equation in an inhomogeneous universe. Our formalism yields the relation between quantities (convergence, shear, and twist) in the gravitational lensing theory and those (rates of expansion, shear and rotation) in the scalar optics theory.

Keywords: cosmology theory — distance measure — optical scalar equations — gravitational lensing

1. Introduction

In a general space, the ray-bundle from a source obeys the optical scalar equations (Sachs 1961)

\[ \begin{align*}
\frac{\partial}{\partial s} + \nabla \cdot E = 0, \\
\frac{\partial}{\partial s} + 2H + \nabla \cdot F = 0,
\end{align*} \]

where \( \Theta, \Sigma, \alpha, \beta, r, \) and \( s \) denote the expansion-, shear-, rotation-rate, and \( s \) is the affine parameter of the null geodesics.

In eq. (1) \( \Sigma \) and \( \alpha \) are the Ricci term and the Weyl term, respectively. The expansion rate is expressed in terms of the cross-sectional area \( A \) or the angular diameter distance \( D \)

\[ D = \frac{1}{3} \sqrt{g} \left( \frac{\partial}{\partial s} + \frac{1}{2} \frac{\partial}{\partial s} \right). \]

Then the first equation of (1) is written as

\[ \frac{\partial}{\partial s} \frac{\partial}{\partial s} + \nabla \cdot E = 0. \]

Homogeneous universe

In a Friedmann-Lemaître universe model (FL model), the angular diameter distance from an observer to a source at a redshift is given by the Matting formula

\[ D_L(z) = \frac{c}{H(z)} \int_0^z \frac{dz'}{H(z')} = \frac{c}{H_0} \left( \frac{1}{1+z} \right) \int_0^z \frac{dz'}{H(z')} = \frac{c}{H_0} \frac{z}{1+z}. \]

The Universe is, on average, described by an FL model, but locally inhomogeneous. The clumpy Universe, on average, described by an FL model, but locally inhomogeneous. The clumpy

2. Multiple Gravitational Lens Effect

The light-ray passing near/through a clump is gravitationally lensed. In this case, the observed flux \( f_{\text{obs}} \) is magnified by factor \( f \) (the gravitational magnification factor). The image is defined as follows:

\[ f_{\text{obs}} = f \int_0^z \frac{dz'}{H(z')} = f D_L(z). \]

In a case of \( f > 1 \), the source of the light-ray is observed as a nearer object. Then the apparent angular diameter distance \( d' \) is given by \( d' = d + \mu'' \delta d_{\text{obs}}(\mu, \Delta u) \).

Now we have a problem whether the distance really satisfies the optical scalar equations for the light-ray passing near/through clumps.

2. Continuous Limit and Optical Scalar Equations

In eq. (11) we take the limit of \( z \rightarrow \infty \), \( \zeta \rightarrow 0 \) (i.e., \( \Delta u \rightarrow -x \), \( x \rightarrow 0 \)) to obtain a second differential equation of \( g \) with respect to \( x \) as follows:

\[ \frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} + \frac{\Delta u}{x^2} = 0. \]

In a case of \( x > 0 \), we can express the gravitational magnification factor as

\[ f = \frac{1}{1+z} \frac{cz}{D_L(z)} = \frac{1}{1+z} \frac{cz}{z}, \]

where

\[ \frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} + \frac{\Delta u}{x^2} = 0. \]

The right hand side of eq. (23) is also rewritten as

\[ \frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} + \frac{\Delta u}{x^2} = 0. \]

The last equation (24) shows that \( d \) is a variable. Moreover, we found that the right hand side of eq. (22) can be regarded as the Ricci term, \( \Omega \). Therefore, the Ricci term obtained in the perturbation theory of the general relativity. The difference between them comes from the thin lens approximation adopted in this presentation. Then we can regard eq. (22) in the Weyl term as long as the approximation is valid.

Finally we obtain the equations of \( \Theta, \Sigma \), and \( \alpha \) as

\[ \frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} + \frac{\Delta u}{x^2} = 0. \]

3. Continuous Limit and Optical Scalar Equations

In a case of \( \Delta u = -x \), \( x \rightarrow 0 \) we can express the gravitational magnification factor as

\[ f = \frac{1}{1+z} \frac{cz}{D_L(z)} = \frac{1}{1+z} \frac{cz}{z}. \]

References


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* This presentation is based on Yoshida, Nakamura & Omote (2004).