Optimal consumption of the finite time horizon
Ramsey problem

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Abstract

In this paper, we study the stochastic Ramsey problem related to an economic
growth model with the CES production function in a finite time horizon. By chang-
ing variables, the Hamilton-Jacobi-Bellman equation associated with this optimiza-
tion problem is transformed. By the viscosity solution technique, we show the exis-
tence of a classical solution of the transformed Hamilton-Jacobi-Bellman equation,
and then give an optimal consumption policy of the original problem.

Key words: Ramsey problem, Hamilton-Jacobi-Bellman equation,
viscosity solutions, economic growth.

1 Introduction

We are concerned with the stochastic Ramsey problem with a finite time horizon $T > 0$ in
order to choose a consumption rate processes $c = \{c(t)\} \in \mathcal{A}_T$ maximizing the discounted
expected utilities with a constant discount rate $\alpha \geq 0$:

$$J_T(c) = \mathbb{E} \left[ \int_0^T e^{-\alpha t} U(c(t)) dt \right],$$

where a controlled process $X^{x,c}$ is governed by the stochastic differential equation (SDE, for short)

$$dX_t = \left[ f(X_t) - c_t \right] dt + \sigma X_t dW_t, \quad t \geq 0, \quad X_0 = x \geq 0;$$

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\(\{W_t\}_{t \geq 0}\) is a standard one-dimensional Brownian motion on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), endowed with a filtration \(\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}\) which is the \(\mathbb{P}\)-augmentation of the filtration generated by the Brownian motion \(W\); \(\sigma > 0\) is a diffusion constant; \(f \in C^1(\mathbb{R}_+)\) is a concave function with \(f(0) = 0\) and \(f'(\infty) < -\infty\); \(\mathcal{A}_T\) denotes the collection of all consumption policies \(c(\cdot)\) which are \(\mathbb{R}_+\)-valued, \(\mathbb{F}\)-progressively measurable processes satisfying \(\int_0^T c(t)dt < \infty\) a.s.;

\[ r^{x,c} = \inf \{t \geq 0 : X^{x,c}(t) = 0\}. \]

We assume that the utility function \(U\) has the following properties:

\[ U \in C(\mathbb{R}_+) \cap C^2(0, \infty), \quad U''(c) < 0 < U'(c), \quad U'(\infty) = U(0) = 0. \tag{1.3} \]

The economic growth model in connection with the Ramsey problem has been studied by [3] and [7]. Recently, [8] treats this problem for the constant elasticity of substitution (CES) production function \(F(z, y)\) with \(\partial F(0^+, y)/\partial z < \infty\) producing the commodity for the capital stock \(z \geq 0\) and the labor force \(y > 0\) in the infinite time horizon case. The capital stock \(z(t)\) and the labor supply \(y(t)\) at time \(t\) are governed by

\[
\begin{align*}
  dy(t) &= \mu y(t)dt - \sigma y(t)dW_t, \quad y(0) = y > 0, \quad \mu \neq 0, \\
  dz(t) &= [F(z(t), y(t)) - \nu z(t) - c(t)y(t)]dt, \quad z(0) = z \geq 0.
\end{align*}
\]

Changing the variables by \(X(t) = z(t)/y(t)\) and \(f(x) = F(x, 1) - (\mu + \nu - \sigma^2)x\) for \(x = z/y\), we observe that the optimal consumption in the growth model can be obtained by a reduction to the stochastic Ramsey problem (1.1)-(1.2). However, [8] contains some incomplete proofs, which will be made correct later, for the existence results on viscosity solutions of the associated Hamilton-Jacobi-Bellman (HJB, for short) equations.

The purpose of this paper is to present a synthesis of optimal consumption policy \(c^* \in \mathcal{A}_T\) for the stochastic Ramsey problem (1.1)-(1.2). The associated HJB equation is given by the parabolic PDE

\[ \alpha u = u_t + \frac{\sigma^2}{2} x^2 u_{xx} + f(x)u_x + \tilde{U}(u_x), \quad t \in (0, T), \quad x > 0, \tag{1.4} \]

\[ u(t, 0) = u(T, x) = 0, \quad t \in [0, T], \quad x \geq 0, \]

where the subscripts denote the partial derivatives and \(\tilde{U}(x)\) is the Legendre transform of \(-U(-x)\), i.e. \(\tilde{U}(x) = \max_{c \geq 0} [U(c) - cx]\), \(x > 0\). The difficulty in solving the problem lies in the fact that (1.4) is degenerate and \(\tilde{U}\) is non-Lipschitz. Changing variables by

\[ v(t, x) = u(T - t, x), \quad t \in [0, T], \quad x \geq 0, \]

we have

\[ \alpha v = -v_t + \frac{\sigma^2}{2} x^2 v_{xx} + f(x)v_x + \tilde{U}(v_x), \quad t \in (0, T), \quad x > 0, \tag{1.5} \]

\[ v(t, 0) = v(0, x) = 0, \quad t \in [0, T], \quad x \geq 0. \]
Our method consists in finding a smooth solution \( v \). By the viscosity solution technique, we show that the transformed HJB equation (1.5) admits a unique solution \( v \), and the restrictive conditions for the existence of \( u \) are relaxed. The optimal consumption rate \( c^* \in \mathcal{A}_T \) can be represented in a feedback form.

The remainder of this paper is organized in the following way: In the next section, we show the existence of viscosity solutions \( v \) of (1.5) with \( T = 1 \). In Section 3, we derive the \( C^{1,2} \)-regularity of \( u \), and we give a synthesis of the optimal consumption \( c^* \in \mathcal{A}_T \) in terms of \( u \). Section 4 is devoted to an application of our results to the infinite time horizon problem.

2 The transformed HJB equations

2.1 Stochastic control problems

Let us consider the stochastic control problem

\[
V(t, x) := \sup_{c \in \mathcal{C}} \mathbb{E} \left[ \int_0^{\tau^{x,c} \wedge T} e^{-\alpha s} U(c_s)ds \right], \quad t, x \geq 0,
\]

where \( \mathcal{C} = \cap_{T > 0} \mathcal{A}_T \). Clearly, \( V(t, 0) = V(0, x) = 0 \). Since \( U \geq 0 \), we note that

the function \( t \mapsto V(t, x) \) is non-decreasing for each \( x \geq 0 \). (2.1)

By the comparison results for solutions of one-dimensional SDEs, see e.g. Proposition 5.2.18 of [4], we see that the mapping \( x \mapsto \tau^{x,c} \) is non-decreasing. Hence we also note

the function \( x \mapsto V(t, x) \) is non-decreasing for each \( t \geq 0 \). (2.2)

For each \( \gamma \in \mathbb{R} \), we denote by \( \mathcal{E}_\gamma(\cdot) \) the exponential martingale

\[
\mathcal{E}_\gamma(t) = \exp \left[ \gamma W_t - \frac{\gamma^2}{2} t \right], \quad t \geq 0.
\]

For \( t, x \geq 0, c \in \mathcal{C} \) and \( \beta \in \mathbb{R} \), by Itô’s formula, we then have

\[
x + (\gamma + \sigma) \int_0^{t \wedge \tau^{x,c}} e^{-(\alpha + \beta)s} \mathcal{E}_\gamma(s)c_sds + \int_0^{t \wedge \tau^{x,c}} e^{-(\alpha + \beta)(t \wedge \tau^{x,c})} \mathcal{E}_\gamma(t \wedge \tau^{x,c})X_{t \wedge \tau^{x,c}}d\mathcal{W}_s
\]

\[
+ \int_0^{t \wedge \tau^{x,c}} e^{-(\alpha + \beta)s} \mathcal{E}_\gamma(s)(\alpha + \beta - \sigma \gamma)X_s - f(X_s))ds.
\]

Since the function \( x \mapsto f(x)/x \) is non-increasing and \( f(0) = 0 \), we note that

\[
l := f'(\infty) \leq \frac{f(x)}{x} \leq f'(0+), \quad x > 0.
\]
Thus, if $\beta \geq f'(0+) - \alpha + \sigma \gamma$, then the local martingale in the left-hand side of (2.3) is super-martingale, and thereby
\[
\mathbb{E} \left[ \int_0^{t \land \tau_{x,c}} e^{-(\alpha + \beta)s} \mathcal{E}_\gamma(s)c_s ds \right] \leq x, \quad t, x \geq 0, \quad c \in \mathcal{C}. \tag{2.5}
\]
In this section and the next section, we set
\[
\gamma := 0 \quad \text{and} \quad \beta := (f'(0+) - \alpha)^+,
\]
and thus (2.5) holds with $\mathbb{E} \gamma(\cdot) \equiv 1$.

**Proposition 2.1** For all $t, x, z \geq 0$ and $y > 0$,
\[
V(t, x) \geq \mathbb{E} \left[ \int_0^t e^{-\alpha s} U(xze^{(l-z)s}), \mathcal{E}_\sigma(s)) ds \right], \tag{2.6}
\]
\[
V(t, x) \leq \bar{U}(y) \frac{1 - e^{-\alpha t}}{\alpha} + xy e^{\beta t}. \tag{2.7}
\]

**Proof** Let $\bar{c}(s) := z\bar{X}(s) = zxe^{(l-z)s} \mathcal{E}_\sigma(s)$. Then, by (2.4),
\[
d\bar{X}(s) = (l - z)\bar{X}(s)ds + \sigma \bar{X}(s)dW_s \leq [f(\bar{X}(s) - \bar{c}(s)) + \sigma \bar{X}(s)dW_s.
\]
Thus the comparison results show $0 < \bar{X} \leq X_{x,c}$ a.e. and $\tau_{x,c} \land t = t$ a.s. which gives (2.6).

Since $\mathcal{E}_\gamma(\cdot) \equiv 1$, by (2.5), we also have
\[
V(t, x) \leq \sup_{c \in \mathcal{C}} \mathbb{E} \left[ \int_0^t e^{-\alpha s} [U(c(s)) - yc(s)] ds \right] + y \sup_{c \in \mathcal{C}} \mathbb{E} \left[ \int_0^t e^{-\alpha s} c(s) ds \right]
\]
\[
\leq \bar{U}(y) \frac{1 - e^{-\alpha t}}{\alpha} + xy e^{\beta t}, \quad y > 0.
\]

**Remark 2.2** We recall that the decreasing convex function $\bar{U}$ on $(0, \infty)$ has the properties
\[
\bar{U}(0+) = U(\infty), \quad \bar{U}(\infty) = U(0) = 0, \quad U(x) = \inf_{y > 0} [\bar{U}(y) + xy], \quad x \geq 0.
\]
Thus we know from (2.6)-(2.7) that
\[
V(t, 0+) = V(0+, x) = 0, \quad t, x \geq 0,
\]
\[
V(t, \infty) = U(\infty) \frac{1 - e^{-\alpha t}}{\alpha}, \quad t > 0.
\]

**2.2 Auxiliary control problems and a dynamic programming principle**

Let us next introduce the auxiliary control problems:
\[
V_n(t, x) := \sup_{c \in \mathcal{C}_n} \mathbb{E} \left[ \int_0^{t \land \tau_{x,c}} e^{-\alpha s} U(c_s) ds \right], \quad t, x \geq 0, \quad n \geq 1, \tag{2.8}
\]
where $\mathcal{C}_n = \{ c \in \mathcal{C} : 0 \leq c \leq n \ \text{a.e.}\}$. 
Let us prove Proposition 2.3.

**Proof.** Fix an arbitrary \( c \in \mathcal{C} \), and set \( c_n := n \wedge c \in \mathcal{C}_n \). Since \( X^{x,c} \leq X^{x,c_n} \) a.e. and \( \tau^{x,c} \leq \tau^{x,c_n} \) a.s., Fatou’s lemma shows

\[
\lim_{n \to \infty} V_n(t,x) \geq \lim_{n \to \infty} \mathbb{E} \left[ \int_0^{\tau^{x,c_n} \land t} e^{-\alpha s} U(c_n(s)) ds \right] \\
\geq \lim_{n \to \infty} \mathbb{E} \left[ \int_0^{\tau^{x,c} \land t} e^{-\alpha s} U(c(s)) ds \right] \geq \mathbb{E} \left[ \int_0^{\tau^{x,c} \land t} e^{-\alpha s} U(c(s)) ds \right],
\]

which means \( \lim_{n \to \infty} V_n(t,x) \geq V(t,x) \). Since \( V_n \leq V \), we have the assertion. \( \square \)

**Lemma 2.4** For \( 0 \leq s \leq t \), \( 0 < x \leq y \) and \( \varepsilon, \xi > 0 \), \( V_n \) satisfies the following properties:

\[
0 \leq V_n(t,x) - V_n(s,x) \leq U(n)(t-s), \quad (2.9)
\]

\[
0 \leq V_n(t,y) - V_n(t,x) \leq t \left[ U(\xi) + \frac{y-x}{y} U'(\xi) \right] n. \quad (2.10)
\]

**Proof.** It is clear that (2.9) holds true. Let \( c \in \mathcal{C}_n \) be arbitrary, and set \( z := x/y \in (0,1) \), \( \hat{c} := zc \) and \( \hat{X} := zX^{y,c} \). Since \( x \mapsto f(x)/x \) is non-increasing,

\[
d\hat{X}_s = [zf(\hat{X}_s/z) - \hat{c}(s)] ds + \sigma \hat{X}_s dW_s \leq [f(\hat{X}_s) - \hat{c}(s)] ds + \sigma \hat{X}_s dW_s.
\]

By the comparison results we have \( zX^{y,c} \leq X^{x,\hat{c}} \) a.e. and \( \tau^{y,c} \leq \tau^{x,\hat{c}} \) a.s. Hence we obtain

\[
\mathbb{E} \left[ \int_0^{\tau^{y,c} \land t} e^{-\alpha s} U(c(s)) ds \right] - V(t,x) \leq \mathbb{E} \left[ \int_0^{\tau^{y,c} \land t} e^{-\alpha s} \{U(c(s)) - U(\hat{c}(s))\} ds \right] \\
\leq U(\xi) \int_0^t e^{-\alpha s} ds + \mathbb{E} \left[ \int_0^{\tau^{y,c} \land t} e^{-\alpha s} \{U(c(s)) - U(zc(s))\} 1_{\{c(s) \geq \xi\}} ds \right] \\
\leq tU(\xi) + (1-z) \cdot \mathbb{E} \left[ \int_0^{\tau^{y,c} \land t} e^{-\alpha s} \cdot c(s) U'(zc(s)) 1_{\{c(s) \geq \xi\}} ds \right] \\
\leq tU(\xi) + t(1-z)nU'(z\xi),
\]

where \( 1 \) is the indicator function. \( \square \)

Next we will establish a dynamic programming principle for \( V_n \).

**Theorem 2.5** The following property holds:

For all \( t, x \geq 0 \) and an \( \mathbb{F} \)-stopping time \( \theta \) with \( \theta \leq t \) a.s.,

\[
V_n(t,x) = \sup_{c \in \mathcal{C}_n} \mathbb{E} \left[ \int_0^{\theta \wedge \tau^{x,c}} e^{-\alpha s} U(c(s)) ds + e^{-\alpha(\theta \wedge \tau^{x,c})} V_n(t-\theta \wedge \tau^{x,c}, X^{x,c}(\theta \wedge \tau^{x,c})) \right]. \quad (2.11)
\]

This property holds true even if \( \theta \) depends on a control \( c \in \mathcal{C}_n \).
Proof Since \( V_n(t,0) = V_n(0,x) = 0 \) and \( \tau^{0,c} = 0 \) a.s., the assertion is trivial if \( t = 0 \) or \( x = 0 \). Fix arbitrary \( t, x > 0 \) and an \( \mathbb{F} \)-stopping time \( \theta \) with \( \theta \leq t \) a.s. Denote \( \tau^{x,c}_s := s \wedge \tau^{x,c} \). Then we observe

\[
V_n(t, x) = \sup_{c \in \mathcal{C}_n} \mathbb{E} \left[ \int_0^{\tau^{x,c}_\theta} e^{-\alpha v} U(c(v)) dv + \mathbb{E} \left[ \int_{\theta}^{\tau^{x,c}_\theta} e^{-\alpha v} U(c(v)) dv \bigg| \mathcal{F}_\theta \right] \cdot \mathbb{1}_{\{\theta < \tau^{x,c}\}} \right]
= \sup_{c \in \mathcal{C}_n} \mathbb{E} \left[ \int_0^{\tau^{x,c}_\theta} e^{-\alpha v} U(c(v)) dv + \mathbb{1}_{\{\theta < \tau^{x,c}\}} \cdot e^{-\alpha \theta} \mathbb{E} \left[ \int_0^{\tau^{x,c}_\theta} e^{-\alpha v} U(\tilde{c}(v)) dv \bigg| z = X^{x,c}(\theta) \right] \right]
\leq \sup_{c \in \mathcal{C}_n} \mathbb{E} \left[ \int_0^{\tau^{x,c}_\theta} e^{-\alpha v} U(c(v)) dv + \mathbb{1}_{\{\theta < \tau^{x,c}\}} \cdot e^{-\alpha \theta} V_n(t - \theta, X^{x,c}(\theta)) \right] \tag{2.12}
= \sup_{c \in \mathcal{C}_n} \mathbb{E} \left[ \int_0^{\tau^{x,c}_\theta} e^{-\alpha v} U(c(v)) dv + e^{-\alpha \theta} V_n(t - \tau^{x,c}_\theta, X^{x,c}(\tau^{x,c}_\theta)) \right],
\]

where \( \tilde{c} \) is the shifted process of \( c \) by \( \theta \), i.e., \( \tilde{c}(s) = c(s + \theta) \).

Let \( \varepsilon > 0 \) and \( c \in \mathcal{C}_n \) be arbitrary. To prove the reverse inequality, we take sequences

\[
0 = t_0 < t_1 < \cdots < t_L = t, \quad 0 = x_0 < x_1 < \cdots < x_M = N,
\]

for large numbers \( L, M, N \geq 1 \), and set

\[
\Delta_t = \max_{1 \leq i \leq L} (t_i - t_{i-1}), \quad \Delta_x = \max_{1 \leq j \leq M - 1} (x_{j+1} - x_j).
\]

Let us define the \( \mathcal{F}_\theta \)-measurable sets \( \{A_{ij}\} \) as

\[
A_{ij} = \{ \omega \in \Omega : \theta \in [t_{i-1}, t_i), \ X(\theta) \in [x_j, x_{j+1}) \}, \quad j = 0, \ldots, M - 1, \\
A_{iM} = \{ \omega \in \Omega : \theta \in [t_{i-1}, t_i), \ X(\theta) \geq N \}, \quad i = 1, \ldots, L.
\]

By means of (2.8), there is a \( \{c_{ij}\} \subset \mathcal{C}_n \) such that

\[
V_n(t - t_i, x_j) - \varepsilon \leq \mathbb{E} \left[ \int_0^{\tau^{x,c}_{t_i,t_j}} e^{-\alpha v} U(c_{ij}(v)) dv \right], \quad 1 \leq i \leq L, \quad 1 \leq j \leq M - 1.
\]

Define \( \tilde{c} \in \mathcal{C}_n \) as

\[
\tilde{c}(s) = c(s) \mathbb{1}_{\{s \leq \theta\}} + \sum_{i=1}^L \sum_{j=1}^{M-1} c_{ij}(s - \theta) \mathbb{1}_{A_{ij}} \mathbb{1}_{\{\theta < \tau^{x,c}\}}.
\]

Then we have

\[
K := \mathbb{E} \left[ \int_0^{\tau^{x,c}_\theta} e^{-\alpha v} U(\tilde{c}(v)) dv - \int_0^{\tau^{x,c}_\theta} e^{-\alpha v} U(c(v)) dv \right]
= \sum_{i=1}^L \sum_{j=1}^{M-1} \mathbb{E} \left[ e^{-\alpha \theta} \mathbb{E} \left[ \int_0^{\tau^{x,c}_{t_i,t_j}} e^{-\alpha v} U(c_{ij}(v)) dv \bigg| z = X^{x,c}(\theta) \right] \right] \mathbb{1}_{A_{ij}} \mathbb{1}_{\{\theta < \tau^{x,c}\}}.
\]
\[ \geq \sum_{i=1}^{M-1} \sum_{j=1}^{M-1} \mathbb{E} \left[ e^{-\alpha t} \mathbb{E} \left[ \int_{t_{i+1}}^{t_j} e^{-\alpha u} U(c_{ij}(v)) dv \right] \mathbbm{1}_{A_{ij}} \cdot \mathbbm{1}_{\{\theta < \tau^{x,c}\}} \right] \]

\[ \geq -\varepsilon + \mathbb{E} \left[ e^{-\alpha t} V_n(t - \theta, X^{x,c}(\theta)) \cdot \mathbbm{1}_{\{\theta < \tau^{x,c}\}} \right] \]

\[ - \sum_{i=1}^{M-1} \sum_{j=1}^{M-1} \mathbb{E} \left[ e^{-\alpha t} \left\{ V_n(t - \theta, X^{x,c}(\theta)) - V_n(t - t_i, x_j) \right\} \mathbbm{1}_{A_{ij}} \cdot \mathbbm{1}_{\{\theta < \tau^{x,c}\}} \right] \]

\[ - \mathbb{E} \left[ e^{-\alpha t} V_n(t - \theta, X^{x,c}(\theta)) \cdot \mathbbm{1}_{\{0 \leq X(\theta) < x_1\}} \right] - \mathbb{E} \left[ e^{-\alpha t} V_n(t - \theta, X^{x,c}(\theta)) \cdot \mathbbm{1}_{\{X(\theta) \geq N\}} \right] \]

\[ =: -\varepsilon + \text{2nd term} - \sum_{i=1}^{M-1} \sum_{j=1}^{M-1} \mathbb{E} \left[ e^{-\alpha t} K_{ij} \right] - K_0 - K_M. \]

Moreover, Lemma 2.4 yields

\[ K_{ij} \leq \left[ V_n(t - t_{i-1}, X^{x,c}(\theta)) - V_n(t - t_i, X^{x,c}(\theta)) \right] \mathbbm{1}_{A_{ij}} \mathbbm{1}_{\{\theta < \tau^{x,c}\}} \]

\[ + \left[ V_n(t - t_i, x_{j+1}) - V_n(t - t_i, x_j) \right] \mathbbm{1}_{A_{ij}} \mathbbm{1}_{\{\theta < \tau^{x,c}\}} \]

\[ \leq U(n) \Delta_t \mathbbm{1}_{A_{ij}} + t \left[ U(x_1) + \frac{\Delta_x}{x_1} U'(\frac{x_1 x_2}{x_{j+1}}) n \right] \mathbbm{1}_{A_{ij}}. \]

Also (2.7) and the standard results on solutions of SDEs with random coefficients give

\[ K_0 \leq V_n(t, x_1) \leq \tilde{t} \tilde{U} \left( \frac{1}{\sqrt{x_1}} \right) + e^{\beta t} \sqrt{x_1}, \]

\[ K_M \leq \mathbb{E} \left[ \mathbbm{1}_{\{\sup_{0 \leq s \leq t} |X^{x,c}(s)| \geq N\}} \left( \tilde{t} \tilde{U}(1) + e^{\beta t} \sup_{0 \leq s \leq t} |X^{x,c}(s)| \right) \right] \]

\[ \leq \left\{ \tilde{t} \tilde{U}(1) + e^{\beta t} \mathbb{E} \left[ \sup_{0 \leq s \leq t} X^{x,c}(s)^2 \right] \right\} \mathbb{P} \left\{ \sup_{0 \leq s \leq t} |X^{x,c}(s)| \geq N \right\} \]

\[ \leq C_0, \]

where \( C_0 = C_0(t, x, n) \) is a positive constant. Combining the above estimates, we obtain

\[ K \geq -\varepsilon + \mathbb{E} \left[ e^{-\alpha t} V_n(t - \theta, X^{x,c}(\theta)) \cdot \mathbbm{1}_{\{\theta < \tau^{x,c}\}} \right] \]

\[ - \left[ \Delta_t \cdot U(n) + \Delta_x \cdot \frac{nt}{x_1} U'(\frac{x_1^2}{N}) + tU(x_1) + \tilde{t} \tilde{U} \left( \frac{1}{\sqrt{x_1}} \right) + e^{\beta t} \sqrt{x_1} + \frac{C_0}{N} \right]. \]

Letting \( \Delta_t, \Delta_x \downarrow 0 \) and then \( x_1 \downarrow 0, N \to \infty \), we have the part \( "\geq" \) of the equality (2.11).

\[ \Box \]

### 2.3 Viscosity solutions

Given a real-valued function \( w \) on \( \mathbb{R}^2_+ \), we shall denote by \( w^* \) (resp. \( w_* \)) its upper (resp. lower) semi-continuous envelope, i.e.

\[ w^*(t, x) = \lim_{\varepsilon \downarrow 0} \sup \left\{ w(s, y) : |t - s| + |x - y| \leq \varepsilon, \quad s, y > 0 \right\}, \quad t, x \geq 0, \quad (2.13) \]

and \( w_* = -(w^*)^\circ \).
**Definition 2.6** Let $\mathcal{G}$ be an $\mathbb{R}$-valued continuous function on $\mathbb{R}_+ \times \mathbb{R} \times (0, \infty) \times \mathbb{R}$, and consider the non-linear PDE

\[
w_t(t,x) + \mathcal{G}(x,w(t,x),w_x(t,x),w_{xx}(t,x)) = 0, \quad t,x > 0.
\]

(2.14)
Assume further that $a \mapsto \mathcal{G}(x,r,q,a)$ is non-increasing. Let $w$ be a locally bounded $\mathbb{R}$-valued function on $(0, \infty)^2$ and $\varphi$ be a smooth $\mathbb{R}$-valued function on $(0, \infty)^2$.

(i) $w$ is called a viscosity super-solution to (2.14) if

\[
\varphi_t(t_0, x_0) + \mathcal{G}(x_0, w(t_0, x_0), \varphi_x(t_0, x_0), \varphi_{xx}(t_0, x_0)) \geq 0
\]

for any local minimizer $(t_0, x_0) \in (0, \infty)^2$ of $(w - \varphi)$ on $(0, \infty)^2$.

(ii) $w$ is called a viscosity sub-solution to (2.14) if

\[
\varphi_t(t_0, x_0) + \mathcal{G}(x_0, w^*(t_0, x_0), \varphi_x(t_0, x_0), \varphi_{xx}(t_0, x_0)) \leq 0
\]

for any local maximizer $(t_0, x_0) \in (0, \infty)^2$ of $(w^* - \varphi)$ on $(0, \infty)^2$.

(iii) $w$ is called a viscosity solution to (2.14) if it satisfies the above requirements (i) and (ii).

Let us define the differential operator $\mathcal{L}$ as

\[
\mathcal{L}\varphi = \frac{\sigma^2}{2} x^2 \varphi_{xx} + f(x)\varphi_x - \alpha\varphi.
\]

**Lemma 2.7** $V_n$ is a continuous viscosity solution to the corresponding HJB equation

\[
(V_n)_t(t,x) - \mathcal{L}V_n(t,x) - \tilde{U}_n((V_n)_x(t,x)) = 0, \quad t,x > 0,
\]

where $\tilde{U}_n(x) = \max_{0 \leq c \leq n} [U(c) - cx]$.

**Proof** By (2.7) and Lemma 2.4 we first know that $V_n$ is continuous on $\mathbb{R}_+^2$. Let $(t_0, x_0) \in (0, \infty)^2$ and $\varphi$ be an $\mathbb{R}$-valued smooth test function on $\mathbb{R}_+^2$.

1. Assume that $0 = (V_n - \varphi)(t_0, x_0) = \min_{\mathbb{R}_+^2} (V_n - \varphi)$. Fix an arbitrary $c \in [0, n]$. For a large number $m$, let us define

\[
\theta_m = \frac{1}{m} \wedge \inf \left\{ s \geq 0 : |X^{x_0,c}(s) - x_0| \geq \frac{x_0}{2} \right\} < \frac{x_0}{2} \wedge \frac{t_0}{2}.
\]

Since $\varphi \leq V_n$, it follows from Theorem 2.5 that

\[
0 = V_n(t_0, x_0) - \varphi(t_0, x_0)
\]

\[
\geq \mathbb{E} \left[ \int_0^{\theta_m} e^{-\alpha s} U(c) ds + e^{-\alpha \theta_m} V_n(t_0 - \theta_m, X^{x_0,c}(\theta_m)) - \varphi(t_0, x_0) \right] 
\]

\[
\geq \mathbb{E} \left[ \int_0^{\theta_m} e^{-\alpha s} U(c) ds + e^{-\alpha \theta_m} \varphi(t_0 - \theta_m, X^{x_0,c}(\theta_m)) - \varphi(t_0, x_0) \right]
\]
Now we use the standard results about solutions of the SDEs to get the estimates

\[
\begin{align*}
\theta_m \leq C_1 \iff 0 \\
\int_0^{\theta_m} e^{-\alpha s} |U(c) - c \cdot \varphi_x(t_0 - s, X^{x_0,c}(s))| ds \\
+ \mathbb{E} \left[ \int_0^{\theta_m} e^{-\alpha s} |\mathcal{L} \varphi(t_0 - s, X^{x_0,c}(s)) - \varphi_t(t_0 - s, X^{x_0,c}(s))| ds \right] \\
\geq \left\{ [U(c) - c \varphi_x(t_0, x_0)] + \mathcal{L} \varphi(t_0, x_0) - \varphi_t(t_0, x_0) \right\} \cdot \mathbb{E} \left[ \int_0^{\theta_m} e^{-\alpha s} ds \right] \\
- c \cdot \mathbb{E} \left[ \int_0^{\theta_m} |\varphi_x(t_0 - s, X^{x_0,c}(s)) - \varphi_x(t_0, x_0)| ds \right] \\
- \mathbb{E} \left[ \int_0^{\theta_m} |\mathcal{L} \varphi(t_0 - s, X^{x_0,c}(s)) - \mathcal{L} \varphi(t_0, x_0)| ds \right] \\
- \mathbb{E} \left[ \int_0^{\theta_m} |\varphi_t(t_0 - s, X^{x_0,c}(s)) - \varphi_t(t_0, x_0)| ds \right] \\
= \left\{ [U(c) - c \varphi_x(t_0, x_0)] + \mathcal{L} \varphi(t_0, x_0) - \varphi_t(t_0, x_0) \right\} \cdot L_0 - c \cdot L_1 - L_2 - L_3.
\end{align*}
\]

Now we use the standard results about solutions of the SDEs to get the estimates

\[
\begin{align*}
0 &\leq \mathbb{E}[1 - m \theta_m] \leq \mathbb{P}\{\theta_m < m^{-1}\} \leq \mathbb{P}\left\{ \sup_{0 \leq s \leq m^{-1}} |X^{x_0,c}(s) - x_0| \geq \frac{x_0}{2} \right\} \leq \frac{C_1}{m}, \\
1 &\geq m \cdot L_0 \geq e^{-\frac{\alpha}{2}} \mathbb{E}[m \theta_m] \geq e^{-\frac{\alpha}{2}} (1 - C_1/m) \xrightarrow{m \to \infty} 1, \\
m \cdot L_1 &\leq m \cdot \mathbb{E} \left[ \int_0^{\theta_m} |\varphi_x(t_0 - s, X^{x_0,c}(s)) - \varphi_x(t_0 - s, x_0)| ds \right] \\
&\quad + m \cdot \mathbb{E} \left[ \int_0^{\theta_m} |\varphi_x(t_0 - s, x_0) - \varphi_x(t_0, x_0)| ds \right] \\
&\leq \sup_{x_0 \leq x \leq 2x_0} |\varphi_x(s, y)| + |\varphi_x(t, y)| \left\{ \mathbb{E} \left[ \sup_{0 \leq s \leq m^{-1}} |X^{x_0,c}(s) - x_0| \right] + \frac{1}{m} \right\} \\
&\leq \frac{C_2}{m} \xrightarrow{m \to \infty} 0,
\end{align*}
\]

where \(C_1 = C_1(x_0, n)\) and \(C_2 = C_2(t_0, x_0, n)\) are positive constants. Similarly, we can prove \(m(L_2 + L_3) \xrightarrow{m \to \infty} 0\). Multiplying (2.15) by \(m\) and sending \(m\) to infinity, we have the super-viscosity property of \(V_n\).

2. Suppose that \(0 = (V_n - \varphi)(t_0, x_0) = \max_{\mathbb{R}_+^2} (V_n - \varphi)\). To prove the sub-viscosity property of \(V_n\), we assume to the contrary that \(-2\varepsilon := -\varphi_t(t_0, x_0) + \mathcal{L} \varphi(t_0, x_0) + U_n(\varphi_x(t_0, x_0)) < 0\). Then there exists a \(\delta > 0\) such that

\[
-\varphi_t(t, x) + \mathcal{L} \varphi(t, x) + \tilde{U}_n(\varphi_x(t, x)) \leq -\varepsilon
\]

for all \((t, x) \in [t_0 - \delta, t_0 + \delta] \times [x_0 - \delta, x_0 + \delta] \subset (0, \infty)^2\). Let \(c \in \mathcal{C}_n\) be arbitrary, and define

\[
\theta^c = \delta \wedge \inf \left\{ s \geq 0 : |X^{x_0,c}(s) - x_0| \geq \delta \right\}, \\
\theta_0 = \min \left[ \inf \left\{ s \geq 0 : X^{x_0,0}(s) \geq x_0 + \delta \right\}, \inf \left\{ s \geq 0 : X^{x_0,n}(s) \leq x_0 - \delta \right\} \right].
\]
By the comparison results, we note $0 < \delta \land \theta_0 \leq \theta^c$ a.s. Since $\varphi \geq V_n$, by Itô’s formula, we obtain
\[
\varphi(t_0, x_0) = \mathbb{E}\left[\int_{0}^{t_{\delta}} \frac{\alpha^2}{2} \frac{\alpha \theta^c}{s} \varphi(t_0 - \theta^c, X_{s, y}(t_0 - \theta^c)) ds \right]
- \mathbb{E}\left[\int_{0}^{t_{\delta}} \alpha \theta^c \varphi_x(t_0 - \theta^c, X_{s, y}(t_0 - \theta^c)) ds \right]
- \mathbb{E}\left[\int_{0}^{t_{\delta}} \alpha \theta^c \varphi(t_0 - \theta^c, X_{s, y}(t_0 - \theta^c)) ds \right]
\leq \mathbb{E}\left[\int_{0}^{t_{\delta}} \alpha \theta^c \varphi(t_0 - \theta^c, X_{s, y}(t_0 - \theta^c)) ds \right] + \mathbb{E}\left[\int_{0}^{t_{\delta}} \alpha \theta^c \frac{\alpha \theta^c}{s} \varphi(t_0 - \theta^c, X_{s, y}(t_0 - \theta^c)) ds \right]
\geq \mathbb{E}\left[\int_{0}^{t_{\delta}} \alpha \theta^c \varphi(t_0 - \theta^c, X_{s, y}(t_0 - \theta^c)) ds \right] + \mathbb{E}\left[\int_{0}^{t_{\delta}} \alpha \theta^c \frac{\alpha \theta^c}{s} \varphi(t_0 - \theta^c, X_{s, y}(t_0 - \theta^c)) ds \right]
\]
By the arbitrariness of $c \in \mathcal{C}_n$ and Theorem 2.5, we see that the above inequality provides the contradiction: $\varepsilon e^{-\delta} \mathbb{E}[\delta \land \theta_0] \leq 0$. \hfill \(\square\)

**Theorem 2.8** \(\mathcal{V}\) is a viscosity solution to the associated HJB equation
\[
\mathcal{V}_t(t, x) - \mathcal{L}\mathcal{V}(t, x) - \tilde{U}(\mathcal{V}_x(t, x)) = 0, \quad t, x > 0 \tag{2.16a}
\]
with the boundary conditions
\[
\mathcal{V}_s(t_0, x_0) = \mathcal{V}_s(t_0, 0) = 0, \quad \mathcal{V}_s^*(t_0, x_0) = \mathcal{V}_s(0, x_0) = 0, \quad t, x > 0. \tag{2.16b}
\]

**Proof** Thanks to Proposition 2.3, it is easy to check
\[
\mathcal{V}_s(t, x) = \lim_{m \to \infty} \inf \left\{ \mathcal{V}_s(n, y) : n \geq m, \ |t - s| + |x - y| \leq m^{-1}, \ s, y > 0 \right\},
\]
\[
\mathcal{V}_s^*(t, x) = \lim_{m \to \infty} \sup \left\{ \mathcal{V}_s(n, y) : n \geq m, \ |t - s| + |x - y| \leq m^{-1}, \ s, y > 0 \right\}.
\]
In view of Lemma 2.7, the viscosity properties follow from the arguments in §6 of [1] because $\tilde{U}_n \nearrow \tilde{U}$ as $n \to \infty$.

Taking (2.1)-(2.2) and (2.7) into account, we see
\[
\mathcal{V}_s^*(t, 0) \leq \lim_{\varepsilon \downarrow 0} \mathcal{V}(t + \varepsilon, 0) \leq \tilde{U}(y) \frac{1 - e^{-\alpha t}}{\alpha} \xrightarrow{y \to 0} 0,
\]
\[
\mathcal{V}_s^*(0, x) \leq \lim_{\varepsilon \downarrow 0} \mathcal{V}(\varepsilon, x + \varepsilon) \leq yx \xrightarrow{y \downarrow 0} 0,
\]
which imply (2.16b). \hfill \(\square\)

**Theorem 2.9** Let \(u\) (resp. \(v\)) be an upper (resp. lower) semi-continuous sub-solution (resp. super-solution) of (2.16a) and
\[
u(t, 0) = v(t, 0) = 0, \quad u(0, x) = v(0, x) = 0, \quad t, x \geq 0, \tag{2.17}
\]
\[
\sup_{t, x \geq 0} \frac{|u(t, x)| + |v(t, x)|}{1 + t + xe^{3t}} < \infty.
\]

(2.18)

Then \( u \leq v \) on \( \mathbb{R}^2_+ \).

**Proof** Assume to the contrary that \( 2\zeta := u(t_0, x_0) - v(t_0, x_0) > 0 \) for some \( t_0, x_0 > 0 \), and let us work towards a contradiction. For any \( m \geq 1 \) and \( \eta > \sigma^2 + 2|f'(0^+)| + 1 \), we define

\[
g(t, x) = (1 + x^2)e^\eta, \\
\varphi(t, x, s, y) = u(t, x) - v(s, y) - \frac{m}{2}(t - s)^2 + (x - y)^2 - \varepsilon[g(t, x) + g(s, y)], \\
M_m = \sup_{\mathbb{R}^2_+} \varphi(t, x, s, y),
\]

where the constant \( \varepsilon > 0 \) is chosen so that

\[
M_m \geq \varphi(t_0, x_0, t_0, x_0) = u(t_0, x_0) - v(t_0, x_0) - 2\varepsilon g(t_0, x_0) = \zeta.
\]

By (2.18), \( M_m = \varphi(t_m, x_m, s_m, y_m) \) for some \( (t_m, x_m), (s_m, y_m) \in \mathbb{R}^2_+ \), and

\[
\zeta + \frac{m}{2}(t_m - s_m)^2 + (x_m - y_m)^2 + \varepsilon[g(t_m, x_m) + g(s_m, y_m)] \leq u(t_m, x_m) - v(s_m, y_m),
\]

which provides that \( \{(t_m, x_m)\}_m \) and \( \{(s_m, y_m)\}_m \) are located in a compact subset of \( \mathbb{R}^2_+ \). Therefore, after passing to a subsequence, \( (t_m, x_m) \xrightarrow{m \to \infty} (t_*, x_*) \in \mathbb{R}^2_+ \). Further, Lemma 3.1 in [1] gives

\[
m[(t_m - s_m)^2 + (x_m - y_m)^2] \xrightarrow{m \to \infty} 0, \\
M_m \xrightarrow{m \to \infty} u(t_*, x_*) - v(t_*, x_*) - 2\varepsilon g(t_*, x_*) \geq \zeta.
\]

Hence we know from (2.17) that \( (t_*, x_*) \in (0, \infty)^2 \) and so \( (t_m, x_m), (s_m, y_m) \in (0, \infty)^2 \) for sufficient large \( m \). By virtue of Theorem 8.3 in [1], there exist \( a, b \in \mathbb{R} \) such that

\[
a x^2 - by^2 \leq 3m(x - y)^2, \quad x, y \in \mathbb{R}, \\
(p_m + \varepsilon g(t_m, x_m), q_m + \varepsilon g_x(t_m, x_m), a + \varepsilon g_{xx}(t_m, x_m)) \in \mathbb{P}^{2,+} u(t_m, x_m), \\
(p_m - \varepsilon g(t_m, x_m), q_m - \varepsilon g_x(t_m, x_m), b - \varepsilon g_{xx}(t_m, x_m)) \in \mathbb{P}^{2,-} v(s_m, y_m),
\]

(2.19)

where \( p_m = m(t_m - s_m), \ q_m = m(x_m - y_m) \) and \( \mathbb{P}^{2,+} w(z) \) (resp. \( \mathbb{P}^{2,-} w(z) \)) is the closed superjet (resp. subjet) of the function \( w \) at the point \( z \), see §8 in [1] for the definitions. By the viscosity properties of the functions \( u \) and \( v \), (2.19) implies that

\[
0 \geq \alpha u(t_m, x_m) + p_m - f(x_m)q_m - \frac{\sigma^2}{2} x_m^2 a - \tilde{U}(q_m + \varepsilon g_x(t_m, x_m)) \\
+ \varepsilon \left[ g(t_m, x_m)g_x(t_m, x_m) - \frac{\sigma^2}{2} x_m^2 g_{xx}(t_m, x_m) \right] \\
\geq \alpha u(t_m, x_m) + p_m - f(x_m)q_m - \frac{\sigma^2}{2} x_m^2 a - \tilde{U}(q_m) + \varepsilon g(t_m, x_m),
\]

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0 \leq \alpha v(s_m, y_m) + p_m - f(y_m)q_m - \frac{\sigma^2}{2} y_m^2 b - \tilde{U}(q_m - \varepsilon g(s_m, y_m))
- \varepsilon \left[ g_t(s_m, y_m) - f(y_m)g_x(s_m, y_m) - \frac{\sigma^2}{2} y_m^2 g_{xx}(s_m, y_m) \right]
\leq \alpha v(s_m, y_m) + p_m - f(y_m)q_m - \frac{\sigma^2}{2} y_m^2 b - \tilde{U}(q_m) - \varepsilon g(s_m, y_m),

where we use the monotonicity of \( \tilde{U} \). Putting these inequalities together, we get

\[
\alpha [u(t_m, x_m) - v(s_m, y_m)] + \varepsilon [g(t_m, x_m) + g(s_m, y_m)]
\leq [f(x_m) - f(y_m)]q_m + \frac{\sigma^2}{2} (x_m^2 a - y_m^2 b)
\leq [f'(0+)] + |f'(\infty)| + 2\alpha^2 \cdot m (x_m - y_m)^2 \xrightarrow{m \to \infty} 0,
\]

which yields the contradiction: \( \alpha \zeta + 2(1 + \alpha)\varepsilon g(x_*, x_*) \leq 0 \). \hfill \Box

**Remark 2.10** By Theorems 2.8&2.9 and (2.7), we have \( V^* \leq V_* \) on \( \mathbb{R}^2_+ \). Since \( V_* \leq V \leq V^* \) on \( \mathbb{R}^2_+ \) by (2.13) and (2.16b), we see that \( V \) is continuous on \( \mathbb{R}^2_+ \).

**Remark 2.11** The results in this section hold true even if \( U(0) > 0 \), exclusive of the continuity of \( V(t, \cdot) \) at \( x = 0 \). Indeed, we can easily prove that

\[
V^*(t, 0+) = V_*(t, 0+) = U(0) + \frac{1}{\alpha} e^{-\alpha t}, \quad t \geq 0.
\]

Replacing \( \tilde{U} \) (resp. \( U \)) with \( \tilde{U}^+ \) (resp. \( U^+ \)) in (2.7) (resp. (2.9)-(2.10)), we don’t require the assumption \( U(0) \geq 0 \) in order to obtain the results in this section. However the concavity of \( V(t, \cdot) \), which is proved for the case \( U(0) = 0 \) in the next section, may be no longer the truth if \( U(0) < 0 \).

### 3 Classical solutions and optimal consumption

In this section, we study the \( C^{1,2} \)-regularity of the continuous viscosity solution \( V(t, x) \) of (2.16), and we present a synthesis of the optimal consumption \( c^* \in \mathcal{A} \) for the optimization problem (1.1).

**Lemma 3.1** The function \( x \mapsto V(t, x) \) is concave for each \( t \geq 0 \), and

\[
\lambda(t_0, t_1, x) := \inf_{\xi \geq t_0 \leq t_1} D^+ \xi V(t, x) > 0, \quad 0 < t_0 \leq t_1, \quad x > 0,
\]

where \( D^+ \xi V \) denotes the right-hand derivative of the concave function \( V(t, \cdot) \).

**Proof** Fix arbitrary \( p \in [0, 1], t, x_1, x_2 > 0 \) and \( c_1, c_2 \in \mathcal{C} \), and define

\[
\tilde{c}_i(s) = c_i(s) \mathbf{1}_{\{s \leq x_i, c_i\}}, \quad i = 1, 2, \quad \tilde{X}(s) = p X^{x_1, \tilde{c}_1} + (1 - p) X^{x_2, \tilde{c}_2}.
\]
Set \( x = px_1 + (1-p)x_2 \) and \( c(s) = p\mathcal{H}_1(s) + (1-p)\mathcal{H}_2(s) \). Thanks to the comparison results, we then have \( \mathcal{X} \leq X^{x,c} \) a.e. and \( \tau^{x_1,\epsilon_1} \lor \tau^{x_2,\epsilon_2} \leq \tau^{x,c} \) a.s. by means of the concavity of \( f \).

Hence the concavity of \( U \) implies

\[
V(t, x) \geq \mathbb{E} \left[ \int_0^{\tau^{x,c} \land t} e^{-\alpha s}U(c(s))ds \right] \\
\geq p \mathbb{E} \left[ \int_0^{\tau^{x_1,\epsilon_1} \land t} e^{-\alpha s}U(c_1(s))ds \right] + (1-p) \mathbb{E} \left[ \int_0^{\tau^{x_2,\epsilon_2} \land t} e^{-\alpha s}U(c_2(s))ds \right],
\]

and thus \( V(t, \cdot) \) is concave.

To prove (3.1), we will assume that \( \lambda(t_0, t_1, x_0) = 0 \) for certain \( 0 < t_0 \leq t_1 \) and \( x_0 > 0 \), and then contradict this assumption. Then for all \( \varepsilon > 0 \) there is an \( s_\varepsilon \in [t_0, t_1] \) such that \( D_x^+ V(s_\varepsilon, x_0) \leq \varepsilon \). Since \( x \mapsto D_x^+ V(s_\varepsilon, x) \) is non-increasing, we see that \( D_x^+ V(s_\varepsilon, x) \leq \varepsilon \) for all \( x \geq x_0 \), and hence

\[
V(s_\varepsilon, x) \leq V(s_\varepsilon, x_0) + \varepsilon (x - x_0), \quad x \geq x_0.
\]

Extracting a subsequence, \( s_\varepsilon \xrightarrow{\varepsilon \downarrow 0} s_0 \in [t_0, t_1] \). Letting \( \varepsilon \downarrow 0 \) in the above inequality, by Proposition 2.1, we have

\[
\mathbb{E} \left[ \int_0^{s_0} e^{-\alpha s} U(x e^{(l-1)\varepsilon \mathcal{X}_s(s)}) ds \right] \leq \tilde{U}(y) \int_0^{s_0} e^{-\alpha s} ds + x_0 e^{\beta s_0} y, \quad x \geq x_0, \quad y > 0.
\]

Letting \( x \to \infty \), Fatou’s lemma gives

\[
U(\infty) \leq \inf_{y > 0} \tilde{U}(y) + x_0 \xi y = U(x_0 \xi), \quad \text{where} \quad \xi^{-1} := e^{-\beta s_0} \int_0^{s_0} e^{-\alpha v} dv.
\]

This is in contradiction with (1.3). \( \square \)

Define

\[
u(t, x) = V(T - t, x), \quad t \in [0, T], \quad x \geq 0. \tag{3.2}
\]

Then we have the first main result.

**Theorem 3.2** Under (1.3), \( u \) satisfies the following properties:

(i) The function \( x \mapsto u(t, x) \) is strictly increasing and concave on \( \mathbb{R}_+ \) for each \( t \in [0, T] \).

(ii) \( u \in C([0, T] \times \mathbb{R}_+) \cap C^{1,2}((0, T) \times (0, \infty)) \) is a solution to the HJB equation (1.4).

**Proof** We have already proved the assertion (i) in the previous lemma. We can also deduce the assertion (ii) from the following:

\[
V \in C(\mathbb{R}_+^2) \cap C^{1,2}((0, \infty)^2) \quad \text{is a solution to (2.16).} \tag{3.3}
\]

To show this claim, let \( Q := (t_0, t_1) \times (x_0, x_1) \) be any bounded open subset of \( \mathbb{R}_+^2 \). We consider the parabolic equation

\[
v_t - \mathcal{L} v - \tilde{U}(v_x \lor \lambda) = 0 \quad \text{in} \quad Q \tag{3.4a}
\]
with the boundary condition
\[ v(t_i, x) = V(t_i, x), \quad v(t, x_i) = V(t, x_i), \quad (t, x) \in \overline{Q}, \quad i = 0, 1, \]

(3.4b)

where the constant \( \lambda = \lambda(t_0, t_1, x_1) > 0 \) as in (3.1).

According to [6], there exists a solution \( v \in C(\overline{Q}) \cap C^{1,2}(Q) \) to (3.4) since \( \tilde{U}(\cdot \cap \lambda) \) is Lipschitz. Clearly, \( v \) is a viscosity solution to (3.4). On the other hand, we can deduce by the concavity of \( V \) that \( \varphi_x(s_0, y_0) \geq D_x^+ V(s_0, y_0) \) for any \( \varphi \in C^{1,2}((0,\infty)^2) \) and \( (s_0, y_0) \in (0,\infty) \) which gives an extremal value of \( V(\cdot - \varphi) \). Therefore we know by Theorem 2.8 that \( V \) is a continuous viscosity solution to (3.4). By the standard arguments as in Theorem 8.2 of [1], we can prove the comparison results on viscosity solutions of (3.4). Hence \( V = v \in C(\overline{Q}) \cap C^{1,2}(Q) \). Thus we have (3.3).

\[ \square \]

Let us define
\[ I(x) = \arg\max_{c \geq 0}[U(c) - cx] = (U')^{-1}(x) \cdot 1_{\{x < U'(0+)\}}, \quad x > 0, \]
\[ G(t, x) = I(u_x(t, x)) \cdot 1_{\{t < T\}} \cdot 1_{\{x > 0\}}, \quad t \in [0, T], \quad x \in \mathbb{R}, \]

and consider the SDE
\[ dX_t^* = \left[f(X_t^*) - G(t, X_t^*)\right] dt + \sigma X_t^* dW_t, \quad t \in [0, T], \quad X_0^* = x \geq 0. \]

(3.5)

**Lemma 3.3** The SDE (3.5) has a unique non-negative solution \( X^* \).

**Proof**

1. In view of (3.2) and (2.6), by L'Hospital's rule and Fatou's lemma, we have
\[
\begin{align*}
\lim_{x \to 0^+} u_x(t, 0+) &\geq \mathbb{E} \left[ \int_0^{T-t} e^{-\alpha s} \lim_{x \to 0} \frac{U(xye^{(l+y)s})}{x} ds \right] \\
&= U'(0+) \frac{y}{\alpha - l + y} \left[ 1 - e^{-\left((\alpha - l + y)(T-t)\right)} \right] \quad \text{as } y \to \infty, U'(0+),
\end{align*}
\]

(3.6)

which implies \( I(u_x(t, 0+)) = 0 \). Hence \( G \) is continuous on \([0, T) \times \mathbb{R} \) and \( x \mapsto G(t, x) \) is non-decreasing.

2. For \( m \geq 1, t \in [0, T] \) and \( x \in \mathbb{R} \), let us define \( g_m(t, x) := G(t, x \vee m^{-1}) - G(t, m^{-1}) \). Then we note that \( g_m(t, \cdot) \) is locally Lipschitz on \( \mathbb{R} \) and \( 0 \leq g_m \leq g_{m+1} \). Since \( f \) is Lipschitz on \( \mathbb{R}_+ \), there exists a unique solution \( x_m \) of the SDE
\[ dx_m(t) = \left[ f(x_m(t)^+) - g_m(t, x_m(t)) \right] dt + \sigma x_m(t) dW_t, \quad t \in [0, T] \]

(3.7)

with an initial data \( x_m(0) = x \geq 0 \). The comparison results show
\[ x_{m+1}(t) \leq x_m(t) \leq x e^{f(0^+) t} \cdot \mathcal{E}_x(t), \quad a.s., \quad t \in [0, T]. \]

Further, by Itô's formula, we notice that
\[ (-x_m(t)^3)^+ = (-x^3)^+ \cdot e^{3\alpha^2 t} \cdot \mathcal{E}_{3\alpha}(t) = 0 \quad a.s., \quad t \in [0, T]. \]
which means $x_m \geq 0$. Now we define $X^*(t) := \lim_{m \to \infty} x_m(t)$, $t \in [0, T]$. Sending $m$ to infinity in (3.7), we know that $X^*$ satisfies (3.5) because $g_m \wedge G$ as $m \to \infty$.

3. Let $X^0$ and $X^1$ be two non-negative solutions of (3.5) and set $Z_t := X^0_t - X^1_t$. Since $G(t, \cdot)$ is non-increasing, by Itô's formula, we have

$$dZ_t^2 = 2Z_t[f(X^0_t) - f(X^1_t) - \{G(t, X^0_t) - G(t, X^1_t)\}] dt + \sigma^2 Z_t^2 dt + 2\sigma Z_t dW_t,$$

where $C_0 = 2(|f'(0+)| + |f'(\infty)|) + \sigma^2$. Hence the comparison results prove that

$$Z_t^2 \leq Z_0^2 \cdot e^{C_0 \tau} \cdot E_2(t) = 0, \quad t \in [0, T]$$

which implies the uniqueness of $X^*$.

Now we obtain another main result.

**Theorem 3.4** Under (1.3), $c^* = \{c^*(t) := G(t, X^*(t))\}_{t \geq 0} \in A_T$ is an optimal consumption process for the optimization problem (1.1).

**Proof** The previous lemma implies $c^* \in A_T$. By the definition of $c^*$ we know that Itô’s formula gives

$$u(0, x) = \mathbb{E} \left[ \int_0^{\theta_n \land \tau} e^{-\alpha t} U(c_t) dt \right] + \mathbb{E} \left[ e^{-\alpha (\theta_n \land \tau)} u(\theta_n \land \tau, X_{\theta_n \land \tau}) \right] \leq J_T(c^*) + \mathbb{E} \left[ e^{-\alpha (\theta_n \land \tau)} u(\theta_n \land \tau, X^{*}_{\theta_n \land \tau}) \right],$$

where $\tau = \tau_x^{c^*}$ and $\theta_n = \inf\{t \geq 0 : X^*(t) \geq n\}$, $n \geq 1$. Since $X^*(t) \leq xe^{f'(0+)}t \cdot E_\sigma(t)$, by (2.7) and Doob’s maximal inequality, we have

$$\mathbb{E} \left[ \sup_{n \geq 1} u(\theta_n \land \tau, X^{*}_{\theta_n \land \tau}) \right] \leq \tilde{U}(1) T + xe^{(\alpha + \beta)T} \mathbb{E} \left[ \sup_{0 \leq t \leq T} E_\sigma(t) \right] < \infty.$$

Letting $n \to \infty$ in (3.8), by the dominated convergence theorem, we obtain $u(0, x) \leq J_T(c^*)$, i.e. $c^*$ is optimal.

By the definition of $u$ in (3.2), we observe that

$$u(t, x) = \sup_{c \in A_T} \mathbb{E} \left[ \int_t^{T \land \theta_{t,c}} e^{-\alpha (s-t)} U(c_s) ds \right], \quad t \in [0, T], \quad x \geq 0,$$

where a controlled process $Y_{t,c}$ is given as a solution to the SDE

$$dY_s = [f(Y_s) - c_s] ds + \sigma Y_s dW_s, \quad s \in [t, T], \quad Y_t = x \geq 0$$

and $\theta_{t,c} = \inf\{s \geq t : Y_{t,c}(s) = 0\}$.

**Theorem 3.5** Assume (1.3). Let $w \in C([0, T] \times \mathbb{R}_+) \cap C^{1,2}([0, T] \times (0, \infty))$ be a solution to the HJB equation (1.4) with polynomial growth. Then $w = u$ on $[0, T] \times \mathbb{R}_+$. 

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Proof Since \( w(t,0) = 0 \) and \( \bar{U}(w_x(t,x)) < \infty \), we first note \( w(t,x) \geq 0 \). Let \((t,x) \in [0,T] \times (0,\infty) \) and \( c \in \mathcal{A}_T \) be arbitrary. 

1. By Itô’s formula and (1.4), we have

\[
e^{-\alpha t} w(t,x) + \sigma \int_t^T e^{-\alpha s} Y_s w_x(s,Y_s) dW_s \geq \int_t^T e^{-\alpha s} U(c_s) ds + e^{-\alpha (T \wedge \theta_t^c)} w(T \wedge \theta_t^c, Y_{T \wedge \theta_t^c}) \geq 0
\]

and thereby

\[
w(t,x) \geq \mathbb{E} \left[ \int_t^{T \wedge \theta_t^c} e^{-\alpha (s-t)} U(c_s) ds \right].
\]

This implies \( w \geq u \).

2. Since \( w \geq u \) and \( w(s,0) = u(s,0) = 0 \), we see that \( w_x(s,0+) \geq u_x(s,0+) \geq U'(0+) \). Thus, by the same line as Lemma 3.3, we know that there exists a unique non-negative solution \( Y^* \) of the SDE

\[
dY_s^* = \left[ f(Y_s^*) - G^w(s,Y_s^*) \right] dt + \sigma Y_s^* dW_s, \quad s \in [t,T], \quad Y_t^* = x \geq 0,
\]

where \( G^w(s,y) = I(w_x(s,y)) \cdot \mathbb{1}_{(t \leq s < T)} \cdot \mathbb{1}_{(y > 0)} \). Since \( w(s,y) \leq C_0(1 + y^k) \) for some constants \( C_0, k > 0 \), the same arguments as in Theorem 3.4 give

\[
w(t,x) \leq \mathbb{E} \left[ \int_t^{T \wedge \theta_t^c} e^{-\alpha (s-t)} U(c^w_s) ds \right] \leq u(t,x),
\]

where \( c^w = \{ c^w(s) := G^w(s,Y^*(s)) \}_{s \geq 0} \in \mathcal{A}_T \) and \( \theta^* = \theta_{t}^{x,c^w} \). Hence the proof is complete. \( \square \)

4 Infinite time horizon Ramsey problem

In this section, we give an application of the above results to the infinite time horizon problem so as to maximize the discounted expected utilities with a discount rate \( \alpha > 0 \):

\[
J_{\infty}(c) = \mathbb{E} \left[ \int_0^{T \wedge \theta_t^c} e^{-\alpha t} U(c(t)) dt \right]
\]

over the class \( \mathcal{C} \). In order to present the similar results to theorems in §3, let us define

\[
\bar{V}(x) = \sup_{c \in \mathcal{C}} \mathbb{E} \left[ \int_0^{T \wedge \theta_t^c} e^{-\alpha t} U(c(t)) dt \right], \quad x \geq 0,
\]

and we make the following hypotheses: There exist \( \beta, \gamma \in \mathbb{R} \) and \( \xi_0 > 0 \) such that

\[
\alpha \geq f'(0+) - \beta + \sigma \gamma, \quad (4.2a)
\]

\[
\bar{F}(\xi_0) := \mathbb{E} \left[ \int_0^\infty e^{-\alpha t} \tilde{U}(\xi_0 e^{-\beta t} \mathcal{E}_s(t)) dt \right] < \infty. \quad (4.2b)
\]
**Theorem 4.1** Let the conditions (1.3) and (4.2) hold true. Then the following assertions are valid:

(i) \( \hat{V} \) is strictly increasing and concave on \( \mathbb{R}_+ \).

(ii) \( \hat{V} \in C(\mathbb{R}_+) \cap C^2(0, \infty) \) is a solution to the HJB equation

\[
-\mathcal{L}\hat{V}(x) - \hat{U}(\hat{V}'(x)) = 0, \quad x > 0, \quad V(0) = 0. \tag{4.3}
\]

**Proof**

1. Let us begin with showing

\[
0 \leq \hat{V}(x) \leq \hat{F}(\xi_0) + x \xi_0, \quad x \geq 0. \tag{4.4}
\]

Let \( x \geq 0 \) and \( c \in \mathcal{C} \) be arbitrary. By (2.5), we have

\[
\mathbb{E}\left[ \int_0^{\tau_{c, \lambda, T}} e^{-\alpha t} U(c_t) dt \right] = \mathbb{E}\left[ \int_0^{\tau_{c, \lambda, T}} e^{-\alpha t} \left( U(c_t) - \xi_0 e^{-\beta T} \mathcal{E}_\gamma(t)c_t \right) dt \right] + \xi_0 \mathbb{E}\left[ \int_0^{\tau_{c, \lambda, T}} e^{-(\alpha + \beta)t} \mathcal{E}_\gamma(t)c_t dt \right]
\]

\[
\leq \mathbb{E}\left[ \int_0^{\infty} e^{-\alpha t} \hat{U}(\xi_0 e^{-\beta T} \mathcal{E}_\gamma(t)) dt \right] + x \xi_0.
\]

Letting \( T \to \infty \), Fatou’s lemma shows (4.4).

Also we can easily deduce the following analogue of (2.6):

\[
\hat{V}(x) \geq \mathbb{E}\left[ \int_0^{\infty} e^{-\alpha t} U(x ze^{(l-z)t} \mathcal{E}_\sigma(t)) dt \right], \quad x, z \geq 0. \tag{4.5}
\]

2. The proof of Lemma 3.1 guarantees the concavity of \( \hat{V} \). We denote by \( \hat{V}'_+ \) the right-hand derivative of \( \hat{V} \). We shall prove that \( \hat{V}'_+(x) > 0 \) for all \( x > 0 \) by the reduction to absurdity. Suppose that \( \hat{V}'_+(x_0) = 0 \) for some \( x_0 > 0 \). By (4.4)-(4.5) and the same line as Lemma 3.1, we then have

\[
\frac{U(\infty)}{\alpha} \leq \hat{F}(\xi_0) + \xi_0 x_0 < \infty.
\]

Hence we may assume \( \hat{U}(0) = U(\infty) < \infty \). Choosing \( \gamma = 0 \) and \( \beta = (f'(0+) - \alpha)^+ \),

\[
\hat{F}(\xi_0) = \int_T^{\infty} e^{-\alpha t} \tilde{U}(\xi_0 e^{-\beta t}) dt + \int_T^{\infty} e^{-\alpha t} \tilde{U}(\xi_0 e^{-\beta T}) dt \leq \frac{1 - e^{-\alpha T}}{\alpha} \tilde{U}(\xi_0 e^{-\beta T}) \leq \frac{1 - e^{-\alpha T}}{\alpha} \tilde{U}(0)
\]

for all \( \xi_0 > 0 \). Combining two inequalities above, we obtain

\[
U(\infty) \leq \inf_{\xi_0 > 0} \left[ \tilde{U}(\xi_0 e^{-\beta T}) + \frac{\alpha}{1 - e^{-\alpha T}} \xi_0 x_0 \right] = U \left( \frac{\alpha e^{\beta T}}{1 - e^{-\alpha T}} x_0 \right).
\]

This is in contradiction with (1.3). Thus the assertion (i) holds true.

3. Since \( \hat{F}(\xi) \searrow 0 \) as \( \xi \to \infty \), we see \( \hat{V}(0+) = 0 \) by (4.4). Hence the concave function \( \hat{V} \).
is continuous on $\mathbb{R}_{+}$.

4. Let us introduce the auxiliary control problems

$$
\hat{V}_n(x) := \sup_{c \in \mathcal{C}_n} \mathbb{E} \left[ \int_0^{\tau_{x,c}^n} e^{-\alpha s} U(c_s) ds \right], \quad x \geq 0, \quad n \geq 1,
$$

and we will establish a dynamic programming principle for $\hat{V}_n$:

For every $\mathcal{F}$-stopping time $\theta$ with $\theta < \infty$ a.s., $V_n$ satisfies the following property:

$$
\hat{V}_n(x) = \sup_{c \in \mathcal{C}_n} \mathbb{E} \left[ \int_0^{\theta \wedge \tau_{x,c}^n} e^{-\alpha s} U(c(s)) ds + e^{-\alpha (\theta \wedge \tau_{x,c}^n)} \hat{V}_n(X_{x,c}(\theta \wedge \tau_{x,c}^n)) \right], \quad x \geq 0, \quad (4.6)
$$

where $\theta$ may depend on a control $c \in \mathcal{C}_n$.

By the same calculation as (2.12), it is clear that the part "$\leq$" in (4.6) holds. Let $0 < t < T$ and $c \in \mathcal{C}_n$ be arbitrary. Theorem 2.5 provides

$$
V_n(T, x) \geq \mathbb{E} \left[ \int_0^{T \wedge \tau_{x,c}^n} e^{-\alpha s} U(c(s)) ds + e^{-\alpha (T \wedge \tau_{x,c}^n)} V_n(T - \theta \wedge \tau_{x,c}^n, X_{x,c}(T \wedge \tau_{x,c}^n)) \right],
$$

where $\tau_{x,c}^n = \tau_{e,c}^n \wedge t$. Since $V_n(T, x) \xrightarrow{T \to \infty} \hat{V}_n(x)$, Fatou's lemma gives

$$
\hat{V}_n(x) \geq \mathbb{E} \left[ \int_0^{\theta \wedge \tau_{x,c}^n} e^{-\alpha s} U(c(s)) ds + e^{-\alpha (\theta \wedge \tau_{x,c}^n)} \hat{V}_n(X_{x,c}(\theta \wedge \tau_{x,c}^n)) \right].
$$

Letting $t \to \infty$, we obtain the part "$\geq$" in (4.6).

5. By the similar arguments to Lemma 2.7, we can prove that $\hat{V}_n$ is a continuous viscosity solution to the associated HJB equation

$$
-\mathcal{L}\hat{V}_n(x) - \hat{U}_n(\hat{V}_n(x)) = 0, \quad x > 0.
$$

Also the same arguments as in Proposition 2.3 ensure that $\hat{V}_n(x) \nrightarrow \hat{V}(x)$ as $n \to \infty$ for each $x \geq 0$. Then Dini's theorem implies that $\hat{V}_n$ converges to $\hat{V}$ locally uniformly on $\mathbb{R}_{+}$. Thus we can deduce from the standard stability results[1, §6] that $\hat{V}$ is a viscosity solution to (4.3).

6. Let $Q := (x_0, x_1)$ be any bounded open subset of $\mathbb{R}_{+}$ and set $\lambda := V_n^*(x_1) > 0$. It is known in [2] that the following boundary value problem has a smooth solution $v$:

$$
-\mathcal{L}v(x) - \hat{U}(v'(x) \vee \lambda) = 0, \quad x \in Q, \quad v(x_i) = \hat{V}(x_i), \quad i = 0, 1. \quad (4.7)
$$

Clearly, $v$ is a viscosity solution to (4.7). On the other hand, since $\hat{V}$ is the concave viscosity solution to (4.3), we see easily that $\hat{V}$ is a continuous viscosity solution to (4.7). Thus, by the comparison results[1, Theorem 3.3] for viscosity solutions, we have $\hat{V} = v \in C(Q) \cap C^2(Q)$. Hence the proof is complete. }

To obtain an optimal consumption process for the infinite time horizon problem, we make the following condition which is slightly stronger than (4.2): There exist $\rho \in (0, \alpha)$, $\beta, \gamma, \epsilon \in \mathbb{R}$ and $\xi_0 > 0$ such that

$$
\alpha - \rho \geq f'(0+) - \beta + \sigma \gamma \quad \text{and} \quad \mathbb{E} \left[ \int_0^\infty e^{-(\alpha - \rho)t} \hat{U}(\xi_0 e^{-\beta t} \mathcal{E}_x(t)) dt \right] < \infty. \quad (4.8)
$$
Theorem 4.2 Assume (1.3) and (4.8). Then $c^* = \{c^*(t) := \hat{G}(X^*(t))\}_{t \geq 0} \in C$ is an optimal policy for the problem (4.1), where $\hat{G}(x) = I(\hat{V}'(x)) \cdot \mathbb{1}_{[x > 0]}$ and $X^*$ is a unique non-negative solution of the SDE

$$dX^*_t = [f(X^*_t) - \hat{G}(X^*_t)] dt + \sigma X^*_t dW_t, \quad t \geq 0, \quad X^*_0 = x \geq 0. \quad (4.9)$$

Proof
1. Taking account of (4.5), the same calculations as in (3.6) provide $\hat{V}'(0+) \geq \hat{U}'(0+)$. Hence we can prove the existence of a unique non-negative solution $X^*$ to (4.9) by the same line as Lemma 3.3, and thereby $c^* \in C$.

2. Let $x \geq 0$ and $c \in C$ be arbitrary. We shall show

$$\lim_{T \to \infty} \mathbb{E} \left[ e^{-\alpha(T \land \tau^x,c)} \hat{V}(X^x,c(T \land \tau^x,c)) \right] = 0. \quad (4.10)$$

To this end, let us define

$$h(x) = \sup_{c \in C} \mathbb{E} \left[ \int_0^{T \land \tau^x,c} e^{-(\alpha - \rho)t} U(c_t) dt \right], \quad x \geq 0.$$

Thanks to Theorem 4.1, we know that $h \in C(\mathbb{R}_+) \cap C^2(0, \infty)$ satisfies

$$0 \leq \hat{V}(x) \leq h(x), \quad \mathcal{L}h(x) + \rho h(x) + \tilde{U}(h'(x)) = 0 < h'(x), \quad x > 0.$$

Thus, by Itô's formula, we have

$$h(x) + \int_0^{T \land \tau^x,c} e^{-\alpha s} X^*_t h'(X^*_t) dW_s = e^{-\alpha (T \land \tau^x,c)} h(X^x,c(T \land \tau^x,c)) + \int_0^{T \land \tau^x,c} e^{-\alpha s} [c_s h'(X^*_s) - \mathcal{L}h(X^*_s)] ds$$

$$\geq \rho \int_0^{T \land \tau^x,c} e^{-\alpha s} h(X^*_s) ds \geq 0.$$

Hence we have

$$\mathbb{E} \left[ \int_0^{\infty} e^{-\alpha s} h(X^*_s) \mathbb{1}_{\{s \leq \tau^x,c\}} ds \right] \leq \frac{h(x)}{\rho} < \infty, \quad (4.11)$$

which implies $\lim_{T \to \infty} \mathbb{E} \left[ e^{-\alpha T} h(X^x,c(T \land \tau^x,c)) \right] = 0$ and thereby (4.10).

3. By the definition of $c^*$ we know that Itô’s formula gives

$$\hat{V}(x) = \mathbb{E} \left[ \int_0^{\theta_n \land T \land \tau} e^{-\alpha t} U(c^*_t) dt \right] + \mathbb{E} \left[ e^{-\alpha (\theta_n \land T \land \tau)} \hat{V}(X^*_n \land T \land \tau) \right]$$

$$\leq J_\infty(c^*) + \mathbb{E} \left[ e^{-\alpha (\theta_n \land T \land \tau)} \hat{V}(X^*_n \land T \land \tau) \right], \quad (4.12)$$

where $\tau = \tau^{x,c^*}$ and $\theta_n = \inf\{t \geq 0 : X^*(t) \geq n\}, \quad n \geq 1$. Since $X^*(t) \leq xe^{f'(0+)t} \cdot \mathcal{E}_\sigma(t)$, by (4.4) and Doob’s maximal inequality, we have

$$\mathbb{E} \left[ \sup_{n \geq 1} \hat{V}(X^*_n \land T \land \tau) \right] \leq \hat{F}(\xi_0) + \xi_0 xe^{f'(0+)T} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathcal{E}_\sigma(t) \right] < \infty.$$

Letting $n \to \infty$ and then $T \to \infty$ in (4.12), by the dominated convergence theorem and (4.10), we obtain $\hat{V}(x) \leq J_\infty(c^*)$, i.e. $c^*$ is optimal as asserted. \hfill $\Box$
Theorem 4.3 Assume (1.3) and (4.2). Let \( w \in C(\mathbb{R}_+) \cap C^2(0, \infty) \) be a solution to the HJB equation (4.3) with polynomial growth. Suppose further that either one of the following conditions holds:

(a) There exist \( \rho_1 > 0, h_0 \in \mathbb{R} \) and \( h \in C(\mathbb{R}_+) \cap C^2(0, \infty) \) such that

\[
\lim_{x \to \infty} \frac{w(x)}{h(x)} < \infty, \quad \mathcal{L}h(x) + \rho_1 h(x) \leq h_0, \quad h(\infty) > 0, \quad h'(x) \geq 0.
\]

(b) \( \lim_{x \to \infty} \frac{U(I(w'(x)))}{w(x)} \geq 2 \rho_2 \) for some \( \rho_2 > 0 \).

Then \( w = \hat{V} \) on \( \mathbb{R}_+ \).

Proof Since \( w(0) = 0 \) and \( \tilde{U}(w'(x)) < \infty \), we first note \( w'(x) \geq 0 \) and \( w(x) \geq 0 \). Let \( x > 0 \) and \( c \in C \) be arbitrary.

1. By the same line as step 1 in the proof of Theorem 3.5, we have

\[
w(x) \geq \mathbb{E} \left[ \int_0^{T_{\Lambda \tau \infty}} e^{-\alpha s} U(c_s) ds \right].
\]

Thus Fatou’s lemma shows \( w \geq \hat{V} \).

2. Since \( w \geq \hat{V} \) and \( \tilde{w}(0) = \hat{V}(0) = 0 \), we see that \( w'(0+) \geq \hat{V}'(0+) \geq U'(0+) \). Thus, by the same line as Lemma 3.3, we know that there exists a unique non-negative solution \( \hat{Y}^* \) of the SDE

\[
dY_t^* = [f(Y_t^*) - \hat{G}^w(Y_t^*)] dt + \sigma Y_t^* dW_t, \quad t \geq 0, \quad Y_0^* = x \geq 0,
\]

where \( \hat{G}^w(y) = I(w'(y)) \cdot 1_{\{y > 0\}} \). Denote \( \tau = \inf\{t \geq 0 : Y^*(t) = 0\} \) and we shall show

\[
\lim_{T \to \infty} \mathbb{E} \left[ e^{-\alpha T} w(Y_T^*) 1_{\{T < \tau\}} \right] = 0.
\]  

(4.13)

2-1. Assume that condition (a) holds. Similarly to (4.11), we have

\[
\mathbb{E} \left[ \int_0^\infty e^{-\alpha s} [h(Y_s^*) - h(0)] 1_{\{s \leq \tau\}} ds \right] \leq \frac{h(x)}{\rho_1} + \frac{(h_0 - \rho_1 h(0))^+}{\rho_1 \alpha} < \infty,
\]

which implies \( \lim_{T \to \infty} \mathbb{E} \left[ e^{-\alpha T} [h(Y_T^*) - h(0)] 1_{\{T \leq \tau\}} \right] = 0 \). We choose a number \( N_1 > 0 \) so that

\[
0 \leq \frac{w(x)}{h(x)} \leq \eta := 1 + \lim_{x \to \infty} \frac{w(x)}{h(x)}, \quad \forall x \geq N_1.
\]

Then we get

\[
\mathbb{E} \left[ e^{-\alpha T} w(Y_T^*) 1_{\{T \leq \tau\}} \right] \leq e^{-\alpha T} [w(N_1) + \eta h(0)^+] + \eta \mathbb{E} \left[ e^{-\alpha T} [h(Y_T^*) - h(0)] 1_{\{T \leq \tau\}} \right] \xrightarrow{T \to \infty} 0.
\]
2-2. Assume that condition (b) holds. We choose a number $N_2 > 0$ so that $U(I(w'(x))) \geq \rho_2 w(x)$ for all $x \geq N_2$. Then Itô’s formula gives

$$w(x) + \sigma \int_0^{T \wedge \tau} e^{-\alpha s} Y_s^w(Y_s^*) dW_s = e^{-\alpha(T \wedge \tau)} w(Y_{T \wedge \tau}^*) + \int_0^{T \wedge \tau} e^{-\alpha s} U(I(w'(Y_s^*))) ds$$

$$\geq \rho_2 \int_0^{T \wedge \tau} e^{-\alpha s} w(Y_s^*) \mathbb{1}_{(Y_s^* \geq N_2)} ds \geq 0,$$

which yields $\lim_{T \to \infty} \mathbb{E} \left[ e^{-\alpha T} w(Y_T^*) \mathbb{1}_{(Y_T^* \geq N_2)} \cdot \mathbb{1}_{(T < \tau)} \right] = 0$, and thereby (4.13).

3. Since $w(y) \leq C_0(1 + y^k)$ for some constants $C_0, k > 0$, by the same arguments as step 3 in the proof of Theorem 4.2, we have

$$w(x) \leq J_\infty(c^{w*}) \leq \widehat{V}(x),$$

where $c^{w*} = \{c^{w*}(t) := \widehat{G}^{w}(Y^*(t)) \}_{t \geq 0} \in \mathcal{C}$. Thus the verification theorem is established.

Here are two examples for the hypotheses (4.2) and (4.8) to be fulfilled.

**Example 4.4 (HARA utilities)** For $a \in (0, 1)$, we consider for the case

$$U(x) = \frac{x^a}{a}, \quad \widetilde{U}(y) = \frac{1 - a}{a} y^\frac{-1}{-a}, \quad x \geq 0, \quad y > 0.$$

Since $\mathbb{E} \left[ \widetilde{U}(\xi e^{-\beta \mathcal{E}(t)}) \right] = \widetilde{U}(\xi) \exp \left( \frac{a}{1-a} \beta + \frac{a}{(1-a)^2} \frac{\gamma^2}{2} \right)$, we need

$$\alpha > A := \frac{a}{1-a} \beta + \frac{a}{(1-a)^2} \frac{\gamma^2}{2}$$

to ensure (4.2b). Let $B := f'(0+) - \beta + \sigma \gamma$. Then we get

$$\min_{\beta, \gamma} (A \vee B) = \min_{\gamma} \left[ \frac{a}{2(1-a)} \gamma^2 + a \sigma \gamma \right] + a f'(0+) = -\frac{a(1-a)}{2} \sigma^2 + a f'(0+),$$

where $\gamma^* = -(1-a)\sigma$ and $\beta^* = (1-a)[f'(0+) - (2-a)\sigma^2/2]$ give the minimum value. Hence (4.2) is equivalent to

$$\alpha > \left[ a f'(0+) - a \frac{1-a}{2} \sigma^2 \right]^+ \quad (4.14)$$

because $\alpha > 0$. Clearly, (4.14) also guarantees (4.8). \(\square\)

**Example 4.5** In addition to (1.3), we assume that

$$U'(0+) = \infty \quad \text{and} \quad AE(U) := \lim_{x \to \infty} \frac{x U'(x)}{U(x)} < 1.$$
By virtue of Lemma 6.3 in [5], we then have
\[ \tilde{U}(\xi) \leq \left( \frac{\xi}{\xi_0} \right)^{\frac{-a}{1-a}} \tilde{U}(\xi_0), \quad 0 < \xi < \xi_0 \]
for certain \( \xi_0 > 0 \) and \( a \in [AE(U), 1) \). Thus we observe
\[
E \left[ \tilde{U}(\xi e^{-\beta t} \mathcal{E}_\gamma(t)) \right] \leq \tilde{U}(\xi_0) \left( \mathbb{P}\{ \xi e^{-\beta t} \mathcal{E}_\gamma(t) \geq \xi_0 \} + E \left[ \left( \frac{\xi e^{-\beta t} \mathcal{E}_\gamma(t)}{\xi_0} \right)^{\frac{-a}{1-a}} \right] \right)
\]
for all \( \xi > 0 \). Since \( \alpha > 0 \), Example 4.4 implies that (4.8) holds true if \( \alpha \) satisfies (4.14).

We conclude this paper with the following remark.

**Remark 4.6**  Here we make the corrections of Theorems 3.3 and 3.4 in the second author’s paper [8].

(i) We need to add the following assumption:

There exists a concave function \( \psi \in \mathcal{B} \cap C^2(0, \infty) \) such that \( \psi(0) = 0 \) and
\[
-\alpha \psi(z) + \frac{\sigma^2}{2} z^2 \psi''(z) + (f(z) - \mu z) \psi'(z) + \tilde{U}(\psi'(z)) \leq 0 < \psi'(z), \quad z > 0.
\]

Then, we replace \( \varphi \) in the estimate of \( J_1 \) in the proof of Lemma 3.2 by \( \psi \) to have
\[
J_1 \leq C_{\varphi} |z - \bar{z}| + \rho[\varphi(z) + \varphi(\bar{z})].
\]

(ii) We replace \( \tau_z \downarrow \theta \) in the proof of Theorem 3.4 by the following: Choosing sufficiently small \( \varepsilon > 0 \), we have
\[
E[\int_0^{\tau_z} e^{-(\beta + 1/\varepsilon)t} c(t) dt] \leq z + E[\int_0^{\tau_z} e^{-(\beta + 1/\varepsilon)t} f(z(t)) dt]
\]
\[
\leq z + f'(0+) E[\int_0^{\infty} e^{-(\beta + 1/\varepsilon)t} q(t) dt] \rightarrow 0 \quad \text{as} \quad z \downarrow 0.
\]

Then, by the concavity of \( U \), we get
\[
E[\int_0^{\tau_z} e^{-(\beta + 1/\varepsilon)t} U(c(t)) dt] \rightarrow 0 \quad \text{as} \quad z \downarrow 0.
\]

By (i) and (ii), we can obtain the same conclusions as these theorems.
References


